Name:
Math 155B Test 2, Thurs 13 Oct 2011, 4 pages, 50 points, 75 minutes.

Median score 40/50, mean score 39.45/50, high score 49/50 (one person).

(7 points) $\int \sqrt{12+4x-x^2}\,dx =$ (Be sure to simplify as much as you can.)

Solution. Use completing the square, and these substitutions

$$\begin{aligned}
12+4x-x^2 &= 4^2-(x-2)^2 && \text{(worth 1 point)}\\
x-2 &= 4\sin\theta && \text{(worth another point)}\\
12+4x-x^2 &= (4\cos\theta)^2\\
\sqrt{12+4x-x^2} &= 4\cos\theta\\
dx &= 4\cos\theta\,d\theta
\end{aligned}$$

(worth 3 points so far, if those last two lines were written correctly). Hence the integral becomes

$$\begin{aligned}
\int (4\cos\theta)(4\cos\theta)\,d\theta &= 16\int(\cos\theta)^2\,d\theta && \text{(worth 4 points)}\\
&= 16\int \frac{1+\cos 2\theta}{2}\,d\theta && \text{(worth 5 points)}\\
&= 8\theta + 4\sin(2\theta)+C = 8\theta + 8\cdot\sin(\theta)\cdot\cos(\theta)+C
\end{aligned}$$

(either of those expressions was worth 6 points)

$$= 8\arcsin\left(\frac{x-2}{4}\right) + 8\cdot\left(\frac{x-2}{4}\right)\cdot\frac{1}{4}\sqrt{12+4x-x^2}+C$$

$$= \boxed{8\arcsin\left(\frac{x-2}{4}\right)+\frac{x-2}{2}\sqrt{12+4x-x^2}+C}$$

This problem apparently was too hard for the test, but it was not beyond your reach. Six students did get an entirely correct answer (i.e., full credit), and several more made errors so minor that they got 6 of the 7 points. One of the answers for which I gave 6 points was

$$8\theta + 4\sin(2\theta) + C \quad = \quad 8\arcsin\left(\frac{x-2}{4}\right) + 4\sin\left[2\arcsin\left(\frac{x-2}{4}\right)\right] + C$$

which is not simplified enough, though it is *equal* to the correct answer. (Keep in mind that $\int \sqrt{12+4x-x^2}\,dx$ is also *equal* to the correct answer.)

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page 2

(5 points) $\displaystyle\int \frac{x^2+4}{(x-1)(x+1)}dx =$

Solution. Begin with long division – i.e., $x^2+4 = 1\cdot(x^2-1)+5$, and so

$$\int \frac{x^2+4}{x^2-1}dx \;=\; \int \frac{(x^2-1)+5}{x^2-1}dx \;=\; \int \left(1+\frac{5}{(x-1)(x+1)}\right)dx$$
$$= \int \left(1+\frac{A}{x-1}+\frac{B}{x+1}\right)dx$$

where we choose A, B so that

$$\frac{A}{x-1}+\frac{B}{x+1} \;=\; \frac{5}{(x-1)(x+1)}$$
$$A(x+1)+B(x-1) = 5$$

When $x=1$, that yields $A = 5/2$; when $x=-1$, that yields $B=-5/2$. Thus the integral becomes

$$\int \left(1+\frac{5/2}{x-1}+\frac{-5/2}{x+1}\right)dx \;=\; \boxed{x+\frac{5}{2}\ln|x-1|-\frac{5}{2}\ln|x+1|+C}$$

which can also be written as

$$\boxed{x+\frac{5}{2}\ln\left|\frac{x-1}{x+1}\right|+C}$$

A common error — several students skipped the long division step, and thought they could find A and B making $A(x+1)+B(x-1)$ equal to x^2+4 – but that can't be done.

page 3

(5 points) $\int_0^{\pi/2} \sin(2x)\cos(x)\,dx =$

Solution, method 1.

$$\begin{array}{rcrcl} \sin(2x+x) &=& \sin(2x)\cos(x) &+& \cos(2x)\sin(x) \\ \sin(2x-x) &=& \sin(2x)\cos(x) &-& \cos(2x)\sin(x) \\ \hline \sin(3x)+\sin(x) &=& 2\sin(2x)\cos(x) && \end{array}$$

and so the problem can be restated as

$$\begin{aligned} \int_0^{\pi/2} \frac{\sin(3x)+\sin(x)}{2}\,dx &= -\Big[\frac{\cos(3x)}{6} + \frac{\cos(x)}{2}\Big]_0^{\pi/2} \\ &= -\Big[\frac{\cos(3\pi/2)}{6} + \frac{\cos(\pi/2)}{2}\Big] + \Big[\frac{\cos(0)}{6} + \frac{\cos(0)}{2}\Big] \\ &= -0-0+\frac{1}{6}+\frac{1}{2} \quad = \boxed{\frac{2}{3}} \end{aligned}$$

Solution, method 2. Use $\sin(2x) = 2\sin(x)\cos(x)$. Substitute $u = \cos x$ and $du = -\sin x\,dx$; the integral becomes

$$\int_0^{\pi/2} 2\cos^2 x \sin x\,dx = -\int_1^0 2u^2\,du = \int_0^1 2u^2\,du = \Big[\frac{2}{3}u^3\Big]_0^1 = \boxed{\frac{2}{3}}$$

(5 points) Three children are sitting on a see-saw. The 90-pound child is on one side, 6 feet from the center (i.e., from the fulcrum). The other two children each weigh 75 pounds, and are on the other side, 4 feet and 5 feet from the center respectively. Where is the center of mass of the three children? I.e., how many feet from the center, and on which side?

Solution. Let's say the 90-pound child is located at -6, and the two 70-pound children are located at $+4$ and $+5$. The total mass is $M = 90 + 75 + 75 = 240$, and the total moment is $(90)(-6) + (75)(4) + (75)(5) = 135$. So the center of mass is located at $135/240 = 9/16$.

$$\boxed{\text{9/16 feet from center, on the side of the two smaller kids}}$$

Arithmetic errors were common, and cost 1 point. Another, slightly less common error was to compute $(90)(6) + (75)(4) + (75)(5) = 1215$ instead of 135, a conceptual error for which I charged 2 points.

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page 4

(5 points) $\displaystyle\int \frac{dx}{1 + x^{2/3}} =$

Solution. Use the rationalizing substitution $u = x^{1/3}$. Then $x = u^3$ and $dx = 3u^2\,du$ (2 points for getting that far), so the integral becomes

$$\begin{aligned} & \int \frac{3u^2\,du}{1+u^2} \qquad \text{(3 points for getting this far)} \\ &= 3\int \frac{(1+u^2)-1}{1+u^2}\,du \;=\; 3\int \left(1 - \frac{1}{1+u^2}\right) du \\ &= 3u - 3\arctan(u) + C \qquad \text{(now we're up to 4 points)} \\ &= \boxed{3\sqrt[3]{x} - 3\arctan(\sqrt[3]{x}) + C} \end{aligned}$$

Some students instead tried the substitution $v = 1+x^{2/3}$, which turns the integral into $\frac{3}{2}\int v^{-1}\sqrt{v-1}\,dv$; or $w = x^{2/3}$, which turns the integral into $\frac{3}{2}\int \frac{\sqrt{w}}{w+1}\,dw$; I gave two points for either of those. But as far as I can see that just makes the problem harder. They then either couldn't finish, or completed the problem by using one computational error or another to transform the problem into a different, more tractable problem. A few got back on track with the second substitution $u = \sqrt{v-1}$, which actually gives us $u = x^{1/3}$.

Another interesting approach was to use the trigonometric identity $1+\tan^2\theta = \sec^2\theta$. Substitute $x^{2/3} = \tan^2\theta$, so $x = \tan^3\theta$, and $dx = 3\tan^2\theta\sec^2\theta\,d\theta$ (worth 2 points if you get this far). The integral becomes

$$\int \frac{3\tan^2\theta\sec^2\theta\,d\theta}{\sec^2\theta} \;=\; 3\int \tan^2\theta\,d\theta \qquad \text{(now we're at 3 points)}$$

$$= \quad 3\int (\sec^2\theta - 1)\,d\theta \quad = \quad 3\tan\theta - 3\theta + C \qquad \text{(now we're at 4 points)}$$

$= 3\sqrt[3]{x} - 3\tan^{-1}(\sqrt[3]{x}) + C$, but no one actually completed this entire computation without any errors.

(4 points) Evaluate $\int x^{-1} \ln x\, dx$.

Solution. This problem really belonged earlier in the semester, not on this test. I put it in to contrast with the next few problem after it.

Substitute $u = \ln x$. Then $du = \frac{1}{x}dx$, and so the integral becomes

$$\int u du \quad = \quad \frac{1}{2}u^2 + C \quad = \quad \boxed{\tfrac{1}{2}(\ln x)^2 + C}$$

Some students apparently confused $\frac{1}{2}\ln(x^2)$, which simplifies to $\ln x$, with $\frac{1}{2}(\ln x)^2$, which does not. Perhaps I should have taken off points from students who omitted the parenthesis altogether, but this time I let it go.

This problem could also be done using integration by parts:

$$I \quad = \quad \int (x^{-1})(\ln x)\,dx \quad = \quad \int (\ln x)'(\ln x)\,dx$$

$$= \quad (\ln x)^2 - \int (\ln x)(\ln x)'\,dx \quad = \quad (\ln x)^2 - \int (\ln x)(x^{-1})\,dx \quad = \quad (\ln x)^2 - I.$$

Adding I to both sides yields $2I = (\ln x)^2 + \text{const}$, and so $I = \frac{1}{2}(\ln x)^2 + C$. But most of the students who attempted this method made some sort of computational error along the way.

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page 5

(5 points) Let b be some constant other than 0. Evaluate $\int x^{b-1} \ln x\, dx$.

(Do *not* choose a particular number for b. Your answer should be a formula involving b, C, and x. And in case you're wondering, the reason I used $b - 1$ rather than b is because it makes the answer come out nicer.)

Solution. Use integration by parts:

$$\int x^{b-1}\ln x\,dx \;=\; \int\Big(\frac{1}{b}x^b\Big)'\ln x\,dx \;=\; \frac{1}{b}x^b\ln x-\int\frac{1}{b}x^b\Big(\ln x\Big)'\,dx$$
$$=\;\frac{1}{b}\left(x^b\ln x-\int x^b\,dx\right)\;=\;\boxed{\frac{x^b}{b}\ln x-\frac{x^b}{b^2}+C}$$

(4 points) Evaluate $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$. (Hint: What value of b should you use?)

Solution. Use $b=1/2$. From the previous problem, we know $\int \frac{\ln x}{\sqrt{x}}dx = (2\ln x-4)\sqrt{x}+C$ (2 points for getting that much). It's an improper integral, since the integrand blows up at $x=0$. Thus we have

$$\int_0^1\frac{\ln x}{\sqrt{x}}\,dx \;=\; \lim_{r\to 0+}\int_r^1\frac{\ln x}{\sqrt{x}}\,dx \;=\; \lim_{r\to 0+}\Big[(2\ln x-4)\sqrt{x}\Big]_r^1$$
$$=\;\lim_{r\to 0+}\Big\{\Big[(0-4)\sqrt{1}\Big]-\Big[(2\ln r-4)\sqrt{r}\Big]\Big\}\;=\;\boxed{-4}$$

where we've used ideas from the section on indeterminate forms to figure that $\lim_{r\to 0+}\sqrt{r}\ln r=0$. (Yes, the answer *should* be negative, because $\ln x<0$ when $0<x<1$.)

If you know what you're doing, you could skip a lot of those steps – you could write

$$\int_0^1\frac{\ln x}{\sqrt{x}}\,dx \;=\; \Big[(2\ln x-4)\sqrt{x}\Big]_0^1 \;=\; \boxed{-4}$$

with the understanding that $(\ln 0)\sqrt{0}$ is an abbreviation for $\lim_{r\to 0+}(\ln r)\sqrt{r}$. I would accept that abbreviation, but I will caution you that some other mathematicians might not.

(4 points) Evaluate $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$.

Solution. Here we use $b=-1$, and so

$$\int\frac{\ln x}{x^2}\,dx \;=\; -\,\frac{1+\ln x}{x}$$

(2 points for getting that much). Then

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{R\to\infty}\int_1^R \frac{\ln x}{x^2}\,dx = \lim_{R\to\infty}\left[-\frac{1+\ln x}{x}\right]_1^R$$

$$= \lim_{R\to\infty}\left\{\left[-\frac{1+\ln R}{R}\right]-\left[-\frac{1+\ln 1}{1}\right]\right\} = \boxed{1}$$

making use of the fact that $\lim_{R\to\infty}\frac{\ln R}{R}=0$. Again, with abbreviations we could write

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \left[-\frac{1+\ln x}{x}\right]_1^\infty = \boxed{1}$$

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page 6

(6 points) Consider the curve $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$, for $1 \le x \le 2$. Set up, but **DO NOT EVALUATE**, the integrals for the following three quantities. In each case, your answer should be of the form $\int_1^2$ [some function of x] dx, simplified as much as possible.

(a) The arclength of the curve.

(b) The surface area, if the curve is revolved around the x-axis.

(c) The surface area, if the curve is revolved around the y-axis.

Solution. Compute $y' = \frac{1}{2}x^2 - \frac{1}{2}x^{-2}$, hence $(y')^2 = \frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{4}x^{-4}$, hence $(y')^2+1 = \frac{1}{4}x^4+\frac{1}{2}+\frac{1}{4}x^{-4}$, and finally $ds/dx = \sqrt{(y')^2+1} = \frac{1}{2}x^2+\frac{1}{2}x^{-2}$. Coming up with $ds/dx = \sqrt{1+(\frac{1}{2}x^2-\frac{1}{2}x^{-2})^2}$ was insufficient simplification, which cost a point.

So the arclength is

$$\int_1^2 \frac{ds}{dx}\,dx = \boxed{\int_1^2 \left(\frac{1}{2}x^2+\frac{1}{2}x^{-2}\right)dx \quad \text{or} \quad \int_1^2 \frac{x^4+1}{2x^2}dx \qquad \text{(a)}}$$

The surface area is $\int_1^2 2\pi r\,ds = \int_1^2 2\pi r \frac{ds}{dx}\,dx$, where r is the radius. In problems (b) and (c), respectively, we have $r = y = \frac{x^3}{6} + \frac{1}{2x}$ and $r = x$. Thus we obtain surface areas

$$\boxed{\int_1^2 2\pi\left(\frac{x^3}{6}+\frac{1}{2x}\right)\left(\frac{1}{2}x^2+\frac{1}{2}x^{-2}\right)dx \quad \text{or} \quad \pi\int_1^2\left(\frac{x^5}{6}+\frac{2x}{3}+\frac{1}{2x^3}\right)dx \qquad \text{(b)}}$$

(either form acceptable), and

$$\boxed{\int_1^2 2\pi x\left(\frac{1}{2}x^2+\frac{1}{2}x^{-2}\right)dx \quad \text{or} \quad \pi\int_1^2\left(x^3+x^{-1}\right)dx \qquad \text{(c)}}$$

Total number of points is 50.