Name (**two** point penalty for omission): **ANSWER KEY**
Math 155B Final exam, Thurs 15 Dec 2011, 8 pages, 75 points, 120 minutes.

Results for this exam —
Class mean, 59.45 points out of 75. That's 79.27%.
Class median, 60 points out of 75; that's 80%.
High score: One person got 75 out of 75.

CIRCLE YOUR ANSWERS

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page 1

(3 points) Find all x satisfying $e^{2x} = 2e^x + 15$.

Solution. That can be rewritten as $e^{2x} - 2e^x - 15 = 0$ and then factored as $(e^x - 5)(e^x + 3) = 0$. Thus we must have $e^x = 5$ or $e^x = -3$. The latter can't happen, since $e^x > 0$ for all x. Thus $x = \boxed{\ln 5}$.

Partial credit: One point for an answer of "$\ln 5$ and $\ln(-3)$."

(5 points) $\int \sec^4 x \ \tan^4 x \, dx =$

Solution. Substitute $u = \tan x$ and $du = \sec^2 x \, dx$ and $1 + u^2 = \sec^2 x$. The integral becomes

$$\int (1+u^2)u^4 \, du \ = \ \int (u^4+u^6) \, du \ = \ \frac{1}{5}u^5 + \frac{1}{7}u^7 + C \ = \ \boxed{\tfrac{1}{5}\tan^5 x + \tfrac{1}{7}\tan^7 x + C}.$$

Common error: Several students got the trigonometric identity relating $\sec x$ and $\tan x$ backwards, and ended up with $\frac{1}{5}\tan^5 x - \frac{1}{7}\tan^7 x + C$, for which I gave 4 points.

page 2

(5 points) Find the arclength of the curve $y = 4 + \frac{2}{3}x^{3/2}$ for $0 \le x \le 8$.

Solution.

$$\begin{aligned} \frac{dy}{dx} &= x^{1/2} \\ \frac{ds}{dx} &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1+x} \\ L = \int ds &= \int_0^8 \sqrt{1+x}\,dx \qquad \text{(worth 3 points)} \\ &= \left[\frac{2}{3}(x+1)^{3/2}\right]_0^8 = \frac{2}{3}(9^{3/2}-1) \qquad \text{(worth 4 points)} \end{aligned}$$

$$= \boxed{\frac{52}{3}} = \boxed{17\frac{1}{3}}.$$

(3 points) Set up, but **do not evaluate**, an integral for the surface area obtained when the curve of the previous problem is rotated around the x-axis.

Solution.

$$A = \int 2\pi r\,ds = \int_0^8 2\pi y \frac{ds}{dx}\,dx = \boxed{\int_0^8 2\pi(4 + \tfrac{2}{3}x^{3/2})\sqrt{1+x}\,dx}.$$

A couple of students tried to write the answer in terms of y instead. I'm not sure why — that makes the problem a little harder — but it can be done that way. We have $x = \left[\frac{3}{2}(y-4)\right]^{2/3}$ and $dx/dy = \left[\frac{3}{2}(y-4)\right]^{-1/3} = \left(\frac{3}{2}y - 6\right)^{-1/3}$. When x goes from 0 to 8, then y goes from 4 to $4 + \frac{32}{3}\sqrt{2}$. Hence

$$A = \int 2\pi r\,ds = \boxed{\int_4^{4+\frac{32}{3}\sqrt{2}} 2\pi y\sqrt{1 + \left(\tfrac{3}{2}y - 6\right)^{-2/3}}\,dy}.$$

page 3

(4 points) $\int \ln x \, dx =$

Solution. Use integration by parts.

$$\int (x)'(\ln x)\, dx \overset{IP}{=} x\ln x - \int (x)(\ln x)'\, dx = \boxed{x\ln x - x + C}.$$

Common error: Omitting the "$+C$" cost 1 point.

(6 points) Find the derivative of $\dfrac{\sqrt[3]{x^2+1}}{e^x\, x^x}$.

Solution. Some students apparently don't know what the symbol $\sqrt[3]{}$ means. It is the **cube root**. That is, $\sqrt[3]{u} = u^{1/3}$. Other interpretations of this symbol produced complete nonsense.

Take the log on both sides.

$$\ln y = \frac{1}{3}\ln(x^2+1) - (x + x\ln x).$$

Some students divided instead of subtracting, rendering all the rest of their work meaningless. A surprisingly common error was to subtract but omit that last pair of parentheses, and thus end up with $-x + x\ln x$ where it should have been $-x - x\ln x$. Two points penalty for that — that's called "loss of invisible parentheses" in the list of Common Errors at

http://www.math.vanderbilt.edu/∼schectex/commerrs/

which has been on your syllabus all semester long.

Differentiate both sides; we obtain

$$\frac{y'}{y} = \frac{2x}{3(x^2+1)} - 1 - 1 - \ln x\,.$$

Multiply both sides through by y, to obtain

$$y' = \boxed{\left\{ \frac{2x}{3(x^2+1)} - 2 - \ln x \right\} \frac{\sqrt[3]{x^2+1}}{e^x\, x^x}}.$$

Unfortunately, some students used a messier and more difficult approach:

$$f = \sqrt[3]{x^2+1} \qquad g = e^x x^x$$

$$f' = \tfrac{2}{3}x(x^2+1)^{-2/3} \qquad g' = e^x x^x(2+\ln x)$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} = \frac{\frac{2}{3}x(x^2+1)^{-2/3}e^x x^x - \sqrt[3]{x^2+1}e^x x^x(2+\ln x)}{e^{2x}x^{2x}}$$

which is worth 5 points. For full credit, you need to simplify it a bit. At the very least, you need to cancel out the common factors of $e^x x^x$:

$$\left(\frac{f}{g}\right)' = \boxed{\frac{\frac{2}{3}x(x^2+1)^{-2/3} - \sqrt[3]{x^2+1}(2+\ln x)}{e^x x^x}}$$

Many students did not know the correct derivative of x^x. That's on the Common Errors list too; two point penalty. Also, some students got the negative of $(f/g)'$ — another two points.

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page 4

(2 points) $\sin^{-1}(\sin(3\pi/4)) =$

Solution. We're looking for a value of θ in $[-\pi/2, \pi/2]$ such that $\sin\theta = \sin(3\pi/4)$. That has answer $\theta = \boxed{\pi/4}$.

No partial credit on this one. If you gave that angle in addition to one or more other angles, you missed the point of the definition of arcsine.

(6 points) $\lim_{x\to 0}(1+\sin x)^{\csc x} =$

Solution. Let $y = (1+\sin x)^{\csc x}$. Then

$$\ln y \;=\; (\csc x)\ln(1+\sin x) \;=\; \frac{\ln(1+\sin x)}{\sin x} \qquad \text{(worth 2 points)}$$

$$\text{hence} \qquad \lim_{x\to 0} \ln y \;\overset{H}{=}\; \lim_{x\to 0} \frac{\frac{\cos x}{1+\sin x}}{\cos x} \;=\; \lim_{x\to 0} \frac{1}{1+\sin x} \;=\; 1$$

where H is l'Hospital's rule. Then $\lim_{x\to 0} y = e^1 = \boxed{e}$.

Alternate method: Substitute $u = \sin x$. We have $u \to 0$ when $x \to 0$, so the problem becomes $\lim_{u\to 0}(1+u)^{1/u}$ which you might recognize to be e, if you've memorized that particular theorem. If you haven't, again take $y = (1+u)^{1/u}$, so $\ln y = \frac{\ln(1+u)}{u}$. Then l'Hospital's rule gives us $\lim_{u\to 0} \ln y = \lim_{u\to 0} \frac{1/(1+u)}{1} = 1$, so $y \to e$.

Many students did not recognize that 1^∞ is an indeterminate form, and they assumed that it behaves like $1^{\text{any power}} = 1$, or else assumed it behaves like $\lim_{u\to\infty} 1.0001^u = \infty$. But when you have

$$(\text{something getting near } 1)^{(\text{something getting big})},$$

its limit depends on which component is going faster.

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page 5

(5 points) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^{3/2}}$ is [convergent] or [divergent] (circle one).

Solution. Use the integral test, and the substitution $u = \ln x$, $du = (1/x)dx$:

$$\int_3^\infty \frac{dx}{x(\ln x)^{3/2}} \;=\; \int_{\ln 3}^\infty u^{-3/2}\,du$$

which is $\boxed{\text{convergent}}$ by the p test.

(6 points) Find the sum of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+5n+6}$

Solution. Using the methods of partial fractions, analyze

$$\frac{1}{n^2+5n+6} = \frac{1}{(n+2)(n+3)} = \frac{1}{n+2} - \frac{1}{n+3} \qquad \text{(worth 2 points)}$$

The summation, then, is equal to

$$\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+\cdots$$

All of the terms cancel out except the $\boxed{1/3}$ at the beginning.

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page 6

(4 points) Let $f(x) = x^5 + x + 1$. Let $g = f^{-1}$. Find $g'(3)$. *Hint:* Do **not** attempt to find a formula for $g(x)$ — you won't be able to, and you don't need to.

Solution. The function f is 1-to-1. You can assume that, because otherwise the problem wouldn't make sense — i.e., g would be undefined. (Or, if you prefer, note that $f'(x) = 5x^4 + 1 > 0$.) By inspection, $f(1) = 3$ and $g(3) = 1$. We calculate $f'(1) = 6$, and therefore $g'(3) = \boxed{1/6}$.

(6 points) $\displaystyle\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} =$

Solution. Some students apparently don't know what the symbol $\sqrt[3]{}$ means. It is the **cube root**. That is, $\sqrt[3]{u} = u^{1/3}$. Other interpretations of this symbol produced complete nonsense.

This problem calls for what our textbook refers to as a "rationalizing substitution." Substitute $u = x^{1/6}$, $x = u^6$, $dx = 6u^5du$ (worth 2 points), to obtain

$$\int \frac{6u^5\,du}{u^3+u^2} = 6\int \frac{u^3\,du}{u+1} \qquad \text{(worth 3 points)}$$

Next, use long division, to obtain

$$u^3 = (u^3+1) - 1 = (u+1)(u^2-u+1) - 1$$

and therefore

$$\begin{aligned} 6\int \frac{u^3\,du}{u+1} &= 6\int\left(u^2-u+1-\frac{1}{u+1}\right)du \qquad \text{(worth 4 points)} \\ &= 2u^3-3u^2+6u-6\ln|u+1|+C \qquad \text{(worth 5 points)} \\ &= \boxed{2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\ln(\sqrt[6]{x}+1)+C}\,. \end{aligned}$$

An alternate approach, if you didn't figure out the long division: Substituting $u+1=v$ and $u=v-1$ yields

$$6\int\frac{u^3\,du}{u+1} = 6\int\frac{(v-1)^3}{v}\,dv = 6\int(v^2-3v+3-v^{-1})\,dv$$

$$= 2v^3-9v^2+18v-6\ln v+C_1 = 2(u+1)^3-9(u+1)^2+18(u+1)-6\ln(u+1)+C_1$$

$$= 2u^3-3u^2+6u-6\ln(u+1)+C_2$$

and then continue as in the previous paragraph.

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page 7

(4 points) Find the derivative of $f(x)=\cos(e^{\sqrt{x}})$.

Solution. Use the chain rule.

$$\frac{d\cos(e^{\sqrt{x}})}{dx} = \frac{d\cos(e^{\sqrt{x}})}{de^{\sqrt{x}}}\cdot\frac{de^{\sqrt{x}}}{d\sqrt{x}}\cdot\frac{d\sqrt{x}}{dx}$$

$$= -\sin(e^{\sqrt{x}})\cdot e^{\sqrt{x}}\cdot\frac{1}{2}x^{-1/2} = \boxed{-\frac{e^{\sqrt{x}}\sin(e^{\sqrt{x}})}{2\sqrt{x}}}$$

(5 points) $\displaystyle\int\frac{dx}{\sqrt{-x^2+6x-8}} =$

Solution. First complete the square. We have $x^2-6x+8=(x-3)^2-1$. Substitute $u=x-3$ to get

$$\int\frac{dx}{\sqrt{-x^2+6x-8}} = \int\frac{du}{\sqrt{1-u^2}} = \arcsin u + C = \boxed{\arcsin(x-3)+C}$$

A surprisingly large number of students seemed to think that $\sqrt{1-(x-3)^2}$ is equal to $-\sqrt{(x-3)^2-1}$. It's not. But in any case, if you worked with the latter, here's what you'd get: Substitute $x-3=\sec\theta$, $dx=\sec\theta\tan\theta\,d\theta$, and $(x-3)^2-1=\tan^2\theta$, so

$$\int\frac{dx}{\sqrt{(x-3)^2-1}} = \int\frac{\sec\theta\tan\theta\,d\theta}{\tan\theta} = \int\sec\theta\,d\theta$$

$$= \ln|\sec\theta+\tan\theta|+C = \ln\left|x-3+\sqrt{(x-3)^2-1}\right|+C$$

I'll give 3 points for that last expression, or fewer points for approximations to it.

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page 8

(5 points) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\sin\left(\frac{1}{n}\right)$ is (circle one)

[absolutely convergent] or [conditionally convergent] or [divergent]

Solution. $\boxed{\text{conditionally convergent}}$. Full credit for that answer, even if no explanation is given. Partial credit for other answers, if some work is shown —

The given series converges, by the alternating series test, since $\sin(\frac{1}{n})$ is decreasing to 0. Mentioning that fact, or showing some understanding of it, was worth 2 points.

The series $\sum\sin(\frac{1}{n})$ diverges, by comparison with the harmonic series $\sum\frac{1}{n}$. That's another two points.

Putting together those facts, and understanding the definitions of the different kinds of convergences, and drawing the right conclusion, is the fifth point.

(6 points) Find the sum of the series $\displaystyle\sum_{n=0}^{n}(2x+3)^n$, and also its interval of convergence.

Solution. You should know that

$$a + ar + ar^2 + ar^3 + \cdots = \begin{cases} \dfrac{a}{1-r} & \text{if } |r| < 1 \\ \text{divergent} & \text{if } |r| \geq 1 \end{cases}$$

In this case we have $a = 1$ and $r = 2x + 3$. Thus the sum is $\boxed{\dfrac{1}{2x+2}}$ (worth 2 points), and the interval of convergence is

$$|2x + 3| < 1 \qquad \text{(worth 2 points)}$$

$$-1 < 2x + 3 < 1$$

$$-4 < 2x < -2$$

$$-2 < x < -1$$

$$\boxed{(-2, -1)}$$

worth 4 points: each round parenthesis (not a square bracket) was worth a point, and each of the two numbers was worth a point.

Total number of points is 75 out of 75.