Section ______ Name (please PRINT):
Math 140 Final Exam, Fall 2009, 6 pages, 50 points, 120 minutes.

(6 points) Find the equation of the tangent line to the graph of

$$(x-y-1)^3 = x$$

at the point $(1,-1)$. *Hint*: You may use any method you like, but I think this problem is easiest if you use implicit differentiation.

Solution. This is page 224 problem 34, though I added the hint. Differentiate both sides of the equation, with respect to x:

$$\left((x-y-1)^3\right)' \;=\; (x)'.$$

Keep in mind that, for any function u, we have $(u^3)' = 3u^2u'$. Thus

$$\underline{3(x-y-1)^2(x-y-1)' \;=\; 1}.$$

Getting that far correctly should be worth 2 points. Rewrite that as

$$\underline{3(x-y-1)^2(1-y') \;=\; 1};$$

that's worth 3 points.

Some students at this point probably will solve for y', but if you do that next, you're doing this the hard way. It's much easier if you now plug in $x = 1$ and $y = -1$:

$$\underline{3(1-(-1)-1)^2(1-y') \;=\; 1}$$

Now we're up to 4 points. And *now* solve for y':

$$3(1)^2(1-y') \;=\; 1$$

$$3(1-y') \;=\; 1$$

$$1-y' \;=\; 1/3$$

$$\underline{y' \;=\; 2/3}$$

That's worth 5 points; that's the slope of the tangent line. So we need a line through $(1,-1)$ with slope $2/3$. That is

$$\boxed{y+1 = (2/3)(x-1)} \quad \text{or} \quad \boxed{y = \tfrac{2}{3}x - \tfrac{5}{3}} \quad \text{or} \quad \boxed{2x-3y=5}.$$

One point for any straight line, or two points for any straight line that goes through $(1, -1)$.

For the students doing it the hard way:

$$3(x-y-1)^2(1-y') \;=\; 1$$

$$1 - y' = \frac{1}{3(x-y-1)^2}$$

$$y' = 1 - \frac{1}{3(x-y-1)^2}$$

4 points for getting that far correctly. Just for reference, if you multiply it all out, that expression is

$$y' \;=\; \frac{3x^2 - 6xy + 3y^2 - 6x + 6y + 2}{3x^2 - 6xy + 3y^2 - 6x + 6y + 3}$$

Common errors:

Some students, in their very first step, differentiated x and got 0 instead of 1. That results in an answer of $y = x - 2$, for which I gave 4 points.

Some students decided that $\big((x-y-1)^3\big)' = (x' - y' - (1)')^3$. Any student who writes something like that ought to flunk the entire course just for that one mistake, but all I really felt able to do was give zero credit on this one problem.

An afterthought. This problem was from the section of the book on implicit differentiation, and so I thought that was the method to use; that's why I gave that hint. But after I had printed up the tests, it occurred to me that implicit differentiation is *not* needed for this problem — in fact, the problem is easier without it! Here's how:

$$(x-y-1)^3 = x$$

Take the cube root on both sides:

$$x - y - 1 = x^{1/3}$$

Now differentiate both sides:

$$1 - y' + 0 = \frac{1}{3}x^{-2/3}$$

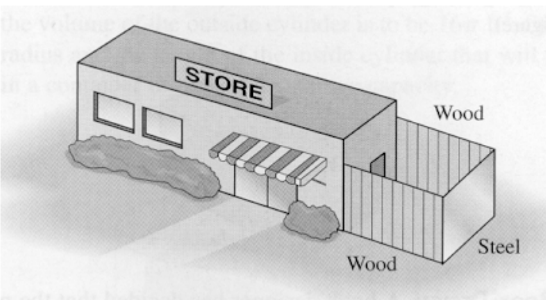
Now plug in $x = 1$, $y = -1$ (actually the latter number isn't needed):

$$1 - y' + 0 = \frac{1}{3}$$

and now solve for $y' = 2/3$ and proceed as in the solution described earlier.

(8 points) The management of the UNICO department store has decided to enclose a rectangular region, 800 square foot in area, outside the building, for displaying potted plants and flowers. One side will be formed by the external wall of the store, two sides will be constructed of pine (wooden) boards, and the fourth side will be made of galvanized steel fencing. If the pine board fencing costs \$6 per running foot, and the steel fencing costs \$3 per running foot, determine the dimensions of the enclosure that can be erected at minimum cost. See figure.

Hints: A "running foot" means a foot along the ground, i.e., the measurement in the horizontal direction; it means that the wooden and steel fences have some fixed height that we're not being told and that we don't need to consider in our calculations. And the external wall of the store is already there, so it doesn't add anything to the cost.



Solution. This is page 319 problem 5. Let W be the length of each of the wooden sides, and let S be the length of the steel side. Then the area is $\underline{WS = 800}$ (that equation is worth 1 point), and the cost is $\underline{C = 6W + 3S + 6W}$ dollars (that equation is worth 3 points), and we want to find W and S to minimize C.

Simplify the formula for C; it is $C = 12W + 3S$.

We can eliminate either variable.

If we eliminate W, that's $W = 800S^{-1}$, and so $C = 12 \cdot 800S^{-1} + 3S$. Thus

$$\underline{C(S) \;=\; 9600S^{-1} + 3S} \qquad \text{(worth 5 points)}$$

$$\underline{C'(S) \;=\; -9600S^{-2} + 3} \qquad \text{(worth 6 points)}$$

$$\text{(optional)} \qquad C''(S) \;=\; 19200S^{-3}.$$

Finding the second derivative is optional. The fact that the second derivative is positive tells us that the function $C(S)$ is concave upward, and so its only relative extremum is a relative minimum — but we could figure that out in other ways as well; it doesn't have to be done using the second derivative. Anyway, the extremum occurs where the first derivative is 0:

$$C'(S) \;=\; 0$$

$$-9600S^{-2} + 3 \;=\; 0$$

$$3 \;=\; 9600S^{-2}$$

$$S^2 \;=\; 3200$$

$$\underline{S \;=\; \sqrt{3200} \;=\; 40\sqrt{2}} \qquad \text{(worth }\underline{\text{7 points}}\text{)}$$

$$W \;=\; 800S^{-1} \;=\; 800/(40\sqrt{2}) \;=\; 20/\sqrt{2} \;=\; 10\sqrt{2}$$

Thus

The steel side is $\sqrt{3200} = 40\sqrt{2} \approx 56.568$ feet; each of the wooden sides is $\sqrt{200} = 10\sqrt{2} \approx 14.142$ feet.

Some students calculated the cost; it is

$$C \;=\; 12W + 3S \;=\; 240\sqrt{2} \;\approx\; \boxed{339.41 \text{ dollars}}.$$

I gave full credit for that answer; but generally I was not able to figure out the causes of errors for answers different from that, and thus could not assign much partial credit.

Alternatively, if we eliminate S, that's $S = 800W^{-1}$, and so $C = 12W + 3 \cdot 800W^{-1}$. Thus

$$\underline{C(W) \;=\; 12W + 2400W^{-1}} \qquad \text{(worth }\underline{\text{5 points}}\text{)}$$

$$\underline{C'(W) \;=\; 12 - 2400W^{-2}} \qquad \text{(worth }\underline{\text{6 points}}\text{)}$$

$$C''(W) \;=\; 2400W^{-3}.$$

Again, finding the second derivative is optional; noting that it is positive may be the easiest way to check that the place where the first derivative vanishes is indeed the number we're looking for. That number is at

$$C'(W) \;=\; 0$$

$$12 - 2400W^{-2} \;=\; 0$$

$$12 = 2400W^{-2}$$
$$W^2 = 200$$
$$\underline{W = 10\sqrt{2}} \qquad \text{(worth } \underline{\text{7 points}}\text{)}$$
$$S = 800W^{-1} = 800/(10\sqrt{2}) = 40\sqrt{2}$$

and thus the same answers.

Some students (for reasons not known to me) disregarded the costs of the wooden boards and the steel fencing, and treated the building materials as though they cost the same per running foot. Thus the cost was simply the perimeter of the three sides of the rectangle — that is, we must minimize $P = S + 2W$ subject to the constraint that $WS = 800$. Then $W = 800S^{-1}$, and so $P(S) = S + 1600S^{-1}$, and $P'(S) = 1 - 1600S^{-2}$. That vanishes at $S = 40$, and then $W = 20$, with a resulting cost of \$360. For that answer I gave 4 points.

(5 points) Find the area of the region completely enclosed by the graphs of the functions $f(x) = \sqrt{x}$ and $g(x) = x^2$. Answer: [ ]

Solution. This is page 461 problem 40. First we must find the places where the two curves intersect, i.e., where $f(x) = g(x)$. That is, $\sqrt{x} = x^2$. That yields two cases:

(i) $x = 0$ is a solution.

(ii) Suppose $x \neq 0$. Then $\sqrt{x} \neq 0$. So from the equation $\sqrt{x} = x^2$ we get

$$\frac{\sqrt{x}}{\sqrt{x}} = \frac{x^2}{\sqrt{x}}$$
$$1 = x^{3/2}$$
$$1 = x.$$

Thus the only intersections of the two curves are at $\underline{x = 0 \text{ and } x = 1}$; establishing that is worth <u>2 points</u>.

So the area we're looking for is $\left|\int_0^1 [f(x) - g(x)]\,dx\right| = \underline{\left|\int_0^1 (\sqrt{x} - x^2)dx\right|}$; getting that far was worth 3 points.

We're going to need to know that $\int (\sqrt{x} - x^2)dx = \frac{2}{3}x^{3/2} - \frac{1}{3}x^3 + C$; writing that expression somewhere is worth 1 point in itself.

Let us compute

$$\int_0^1 (\sqrt{x} - x^2)dx = \underline{\left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1} \qquad \text{(worth } \underline{4 \text{ points}}\text{)}$$

$$= \frac{2}{3} - \frac{1}{3} = \boxed{\frac{1}{3}} \quad \text{or} \quad \boxed{0.333\cdots}.$$

(5 points) $\displaystyle\int_4^\infty \frac{2}{x^{3/2}}dx =$

Solution. This is page 518 problem 17.

$$\lim_{R\to\infty} \int_4^R \frac{2}{x^{3/2}}dx \quad \text{(worth 1 point)}$$

$$= \lim_{R\to\infty} \int_4^R 2x^{-3/2}dx \quad \text{(worth 2 points)}$$

$$= \lim_{R\to\infty} \left[-4x^{-1/2}\right]_4^R \quad \text{(worth 3 points)}$$

$$= \lim_{R\to\infty} \left[\frac{-4}{\sqrt{R}} + \frac{4}{\sqrt{4}}\right] \quad \text{(worth 4 points)}$$

$$= \boxed{2}.$$

(5 points) Sketch the graph of the function

$$g(x) = (x + 2)^{3/2} + 1.$$

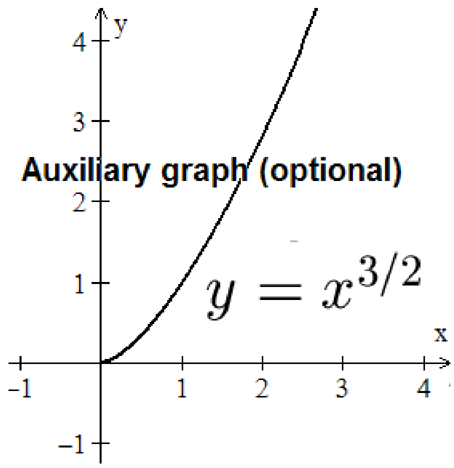
If it has any of these features, label them in your graph: x-intercepts, y-intercept, horizontal and/or vertical asymptotes, relative maxima and minima, inflection points. Also, a glance at your sketch should make it evident where the function is defined, and where it is increasing, decreasing, concave upward, or concave downward.

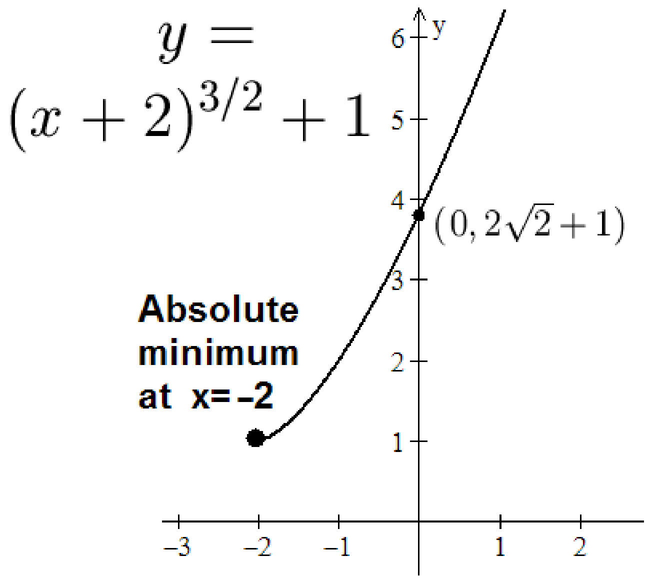
Solution. This is page 293 problem 60. You could get this curve by starting with the picture for $y = x^{3/2}$ (if you know that one), and shifting it two units to the left and one unit up.

So let's describe $y = x^{3/2}$ first. It is domained only for $0 \le x < \infty$, and it is nonnegative there; it has an absolute minimum at $(0, 0)$. Its first derivative is $y' = \frac{3}{2}x^{1/2}$, which is also defined for $x \ge 0$ and positive there, so y is increasing. The second derivative is $y'' = \frac{3}{4}x^{-1/2}$, which is defined and positive for $x > 0$, so y is concave upward. Optional additional information: Since y' vanishes at $x = 0$, it has a horizontal tangent there.

Similarly, $g(x)$ is defined on the interval $[-2, \infty)$, and it is increasing and concave upward on that interval. It has an absolute minimum at $(-2, 1)$. Thus it has no x-intercept. We find its y-intercept, $(0, 2\sqrt{2}+1)$, by calculating $g(0) = 2^{3/2} + 1 \approx 3.828$. Optional: Horizontal tangent at $x = -2$.

Deduct a point for any false assertions about the graph — e.g., the existence of an asymptote, a maximum, an x-intercept, an inflection point.





(6 points) On the interval $0 \le x \le 2$, the function $f(x) = x\sqrt{4 - x^2}$ has

$$\begin{pmatrix} \text{absolute} \\ \text{maximum} \\ \text{value} \end{pmatrix} = \boxed{} \qquad \begin{pmatrix} \text{absolute} \\ \text{minimum} \\ \text{value} \end{pmatrix} = \boxed{}.$$

Solution. This is page 307 problem 38. We need to check the values of $f(x)$ at all the places where $f'(x)$ doesn't exist or $f'(x) = 0$, as well as the endpoints of the interval. It will be slightly easier to differentiate $f(x)$ if we first rewrite it as $f(x) = (4x^2 - x^4)^{1/2}$ (though that is not necessary). Then

$$\begin{aligned} f'(x) &= \tfrac{1}{2}(4x^2 - x^4)^{-1/2}(8x - 4x^3) \\ &= \tfrac{1}{2}(4 - x^2)^{-1/2}(8 - 4x^2) \\ &= \frac{2(2 - x^2)}{\sqrt{4 - x^2}} \end{aligned}$$

Computing $\underline{f'(x)}$ correctly is worth <u>1 point</u>. Now we observe that $f'(x)$ is undefined at $x = \pm 2$ and vanishes at $x = \pm\sqrt{2}$. Thus we need to check at $\underline{x = 0, \sqrt{2}, 2}$; identifying those three numbers brings us up to <u>4 points</u>.

$$f(0) = 0, \qquad f(\sqrt{2}) = 2, \qquad f(2) = 0.$$

The highest of these is 2; the lowest is 0.

$$\boxed{\begin{array}{c}\text{Absolute maximum is 2;} \\ \text{absolute minimum is 0.}\end{array}}$$

(5 points) Find the derivative of $y = x^{\ln x}$. *Hint:* Although you may use any method you like, I think this problem is probably easiest if you use the method of logarithmic differentiation.

Solution. This is problem 50 on page 377. Begin by noting that

$$\ln y \;=\; \ln\left(x^{\ln x}\right) \;=\; (\ln x)(\ln x) \;=\; (\ln x)^2 \qquad \text{(worth 2 points).}$$

Differentiate both sides of that equation; we obtain

$$\frac{y'}{y} \;=\; 2\,(\ln x)\,(\ln x)' \;=\; 2\,(\ln x)\cdot\frac{1}{x} \;=\; \frac{2}{x}\ln x \qquad \text{(worth 4 points).}$$

Now multiply both sides of that equation by y. This yields

$$y' \;=\; \boxed{\frac{2}{x}\,(\ln x)\,x^{\ln x}} \;\text{ or }\; \boxed{2(\ln x)x^{\ln x - 1}}\,.$$

Generally I gave 0 points for students who were clueless — i.e., who used methods that were entirely erroneous and irrelevant.

One of the most common errors is this: Some students thought they could use the formula $\frac{d}{dx}\left(x^k\right) = kx^{k-1}$. But in fact, that formula is only valid when k is a constant — I stressed this in class — and so it is not valid when you plug in $k = \ln x$. Interestingly, however, just by coincidence, if you do plug in $k = \ln x$, the answer that you get is $(\ln x)x^{\ln x - 1}$, which differs from the correct answer by only a factor of 2. However, the fact that this wrong answer so closely resembles the correct answer does *not* mean that it gets a large amount of partial credit. With a quick glance at the student's intermediate steps, I can see whether the student had the right method plus an arithmetic error (deduct only one point), or was completely clueless and treated $\ln x$ as a constant (zero credit).

(5 points) $\displaystyle\int_1^2 \frac{\ln x}{x}\,dx = \boxed{}$

Solution. This is page 488 problem 28. Use the substitution

$$u = \ln x, \qquad du = \frac{1}{x}dx, \qquad \begin{array}{c|c} x & u \\ \hline 2 & \ln 2 \\ 1 & 0 \end{array}$$

Writing that much correctly is worth 3 points. Now

$$\int_1^2 \frac{\ln x}{x}\,dx \;=\; \int_{x=1}^{x=2} (\ln x)\cdot\frac{1}{x}dx \;=\; \int_{u=0}^{u=\ln 2} u\cdot du$$

(now we're up to 4 points)

$$= \left[\frac{1}{2}u^2\right]_0^{\ln 2} = \boxed{\tfrac{1}{2}(\ln 2)^2} \approx \boxed{0.2402\cdots}.$$

A common error: Some students did not change the limits of integration when they did the substitution, and so they ended up with

$$\int_1^2 u du = \left[\frac{1}{2}u^2\right]_1^2 = \frac{1}{2}\left(2^2 - 1^2\right) = \frac{3}{2}$$

for which I gave them 3 points.

Remark. One might be tempted to apply integration by parts. The problem *can* be done that way, but it's tricky. Integration by parts yields

$$\int (\ln x)(x^{-1})dx = \int (\ln x)(\ln x)'dx \overset{IP}{=} (\ln x)^2 - \int (\ln x)'(\ln x)dx = (\ln x)^2 - \int (x^{-1})(\ln x)dx.$$

At first glance, it looks like we've made no improvement at all — the new integral is essentially the same as the integral we started with. But it has this crucial difference: it now has a *minus sign* in front of it. So if we add that integral to both sides of the equation, we get

$$2\int (\ln x)(x^{-1})dx = (\ln x)^2 + C,$$

and finally $\int (\ln x)(x^{-1})dx = \frac{1}{2}(\ln x)^2 + C$. This yields the same answer as the other method mentioned above.

(5 points) (Circle your answer.) $\int xe^{x/4}\,dx =$

Solution. This is page 488 problem 3. We will use integration by parts,

$$(IP) \qquad \int uv'dx = uv - \int uv'dx$$

with

$$u = x, \qquad v' = e^{x/4}, \qquad u' = 1, \qquad v = 4e^{x/4}.$$

Thus

$$\begin{aligned}
& \int (x)\left(e^{x/4}\right) dx \\
= \; & \int (x)\left(4e^{x/4}\right)' dx \\
\stackrel{IP}{=} \; & (x)\left(4e^{x/4}\right) - \int (x)'\left(4e^{x/4}\right) dx \qquad \text{(worth 3 points)} \\
= \; & 4xe^{x/4} - 4\int e^{x/4} dx \qquad \text{(worth 4 points)} \\
= \; & \boxed{4xe^{x/4} - 16e^{x/4} + C} = \boxed{4e^{x/4}(x-4) + C}.
\end{aligned}$$

Deduct 1 point for omitting the C.

Total number of points is 50.