Section ______ Name (please PRINT):
Math 140 Final Exam, Fall 2009, 6 pages, 50 points, 120 minutes.

(6 points) Find the equation of the tangent line to the graph of

$$x^2y^3 - y^2 + xy - 1 = 0$$

at the point $(1, 1)$. *Hint*: You may use any method you like, but I think this problem is easiest if you use implicit differentiation.

Solution. This is page 224 problem 33, though I added the hint. Differentiate both sides of the equation, with respect to x:

$$\left(x^2y^3\right)' - \left(y^2\right)' + (xy)' - (1)' = (0)'.$$

Keep in mind that $(uv)' = u'v + uv'$. Thus, we have

$$\left(\left(x^2\right)'\left(y^3\right) + \left(x^2\right)\left(y^3\right)'\right) - \left(y^2\right)' + (x'y + xy') - 0 = 0.$$

Also keep in mind that $(x^k)' = kx^{k-1}$, but $(y^k)' = ky^{k-1}y'$, since we're differentiating with respect to x. Perhaps you can understand that better if you think about the fact that $x' = 1$, and so in fact $(x^k)' = kx^{k-1}x'$ too.

$$\underline{2xy^3 + 3x^2y^2y' - 2yy' + y + xy' = 0}.$$

Getting that far correctly is worth <u>3 of the 6 points</u>.

Some students at this point probably will sove for y', but if you do that next, you're doing this the hard way. It's much easier if you now plug in $x = y = 1$:

$$(2 \cdot 1 \cdot 1^3) + (3 \cdot 1^2 \cdot 1^2 \cdot y') - (2 \cdot 1 \cdot y') + 1 + (1 \cdot y') = 0.$$

$$\underline{2 + 3y' - 2y' + 1 + y' = 0}.$$

Now we're up to <u>4 points</u>. And *now* solve for y':

$$3 + 2y' = 0$$

$$\underline{y' = -3/2}$$

Now we're up to <u>5 points</u>. That's the slope of the tangent line. So we need a line through $(1, 1)$ with slope $-3/2$. That is

$$\boxed{y - 1 = (-3/2)(x - 1)}$$

or $3(x-1)+2(y-1)=0$ or $\boxed{3x+2y=5}$.

One point for any straight line, or two points for any straight line that goes through $(1,1)$.

For students who did it the hard way:

$$2xy^3 + 3x^2y^2y' - 2yy' + y + xy' = 0.$$

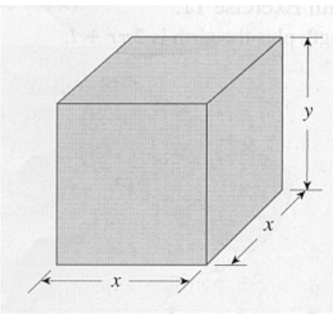
$$(2xy^3 + y) + (3x^2y^2 - 2y + x)y' = 0$$

$$y' = -\frac{2xy^3 + y}{3x^2y^2 - 2y + x}$$

I'll give 4 points for getting that far correctly, or a bit less for getting something close to that.

Students who came up with something that did not resemble that at all, evidently had no understanding of implicit differentiation, and received no credit for it (but still might have one or two points for getting the equation of a line through $(1,1)$).

(8 points) A box is to have a square base and a volume of 20 cubic feet. If the material for the base costs 30 cents per square foot, the material for the sides costs 10 cents per square foot, and the material for the top costs 20 cents per square foot, determine the dimensions of the box that can be constructed at a minimum cost. See diagram.



Solution. This is page 320 problem 10. The base is x feet by x feet, and the height is y feet; thus the volume is $x^2y = 20$ cubic feet.

Considering the areas and costs of the parts of the box:

type	quantity	area	cost/sq.ft.	total cost in cents
base	1	x^2	30 cents	$30x^2$
sides	4	xy	10 cents	$40xy$
top	1	x^2	20 cents	$20x^2$
total				$50x^2 + 40xy$

So we want to find the values of x and y that will minimize $\underline{C = 50x^2 + 40xy}$ (that equation is worth 3 points), subject to the constraint that $\underline{x^2 y = 20}$ (that equation is worth 1 point).

It will be simpler to eliminate y; thus

$$y = 20x^{-2}.$$

(We could instead eliminate x, with the equation $x^2 = 20y^{-1}$ and then $x = \sqrt{20y^{-1}}$, but that would be more complicated. That's worked out later in this solution.)

Substituting $y = 20x^{-2}$ into our formula for C, we get

$$\underline{C(x)} \;=\; 50x^2 + 40x(20x^{-2}) \;=\; \underline{50x^2 + 800x^{-1}}$$

Getting this far correctly is worth 5 points.

$$\underline{C'(x) \;=\; 100x - 800x^{-2}}$$

Now we're up to 6 points.

$$\text{(optional)} \qquad C''(x) \;=\; 100 + 1600x^{-3}.$$

Finding the second derivative is optional, but noting that it is positive is one easy way to tell that $C(x)$ is concave upward, and therefore has a unique minimum at the place where the first derivative vanishes.

Now compute

$$\begin{aligned} C'(x) &= 0 \\ 100x - 800x^{-2} &= 0 \\ 100x &= 800x^{-2} \\ x^3 &= 8 \\ \underline{x = 2} & \qquad \text{(worth 7 points)} \\ y = 20x^{-2} &= 20/4 = 5 \end{aligned}$$

The box has height $y = 5$ feet, and its base is a square whose sides are each $x = 2$ feet long.

Alternatively, if we eliminate x, we get $x = \sqrt{20y^{-1}} = 2\sqrt{5y^{-1}}$, as noted earlier. Then

$$\underline{C = 1000y^{-1} + 80\sqrt{5}y^{1/2}} \qquad \text{(worth } \underline{\text{5 points}}\text{)}$$

$$\underline{C'(y) \;=\; -1000y^{-2} + 40\sqrt{5}y^{-1/2}} \qquad \text{(worth } \underline{\text{6 points}}\text{)}$$

Now set $C'(y)$ equal to 0 and solve for y; we get

$$1000y^{-2} = 40\sqrt{5}y^{-1/2}$$
$$25 = \sqrt{5}y^{3/2}$$
$$5^2 = 5^{1/2}y^{3/2}$$
$$\underline{y = 5} \qquad \text{(worth } \underline{\text{7 points}}\text{)}$$

and then $x = 2\sqrt{5y^{-1}} = 2$.

(5 points) Find the area of the region completely enclosed by the graphs of the functions $f(x) = x^2$ and $g(x) = x^3$. Answer:

Solution. This is page 461 problem 37. First we must find the places where the two curves intersect, i.e., where $f(x) = g(x)$. That is, $x^2 = x^3$. One solution is $x = 0$. When $x \neq 0$, then $x^2 \neq 0$, and so we have

$$x^2 = x^3$$
$$\frac{x^2}{x^2} = \frac{x^3}{x^2}$$
$$1 = x.$$

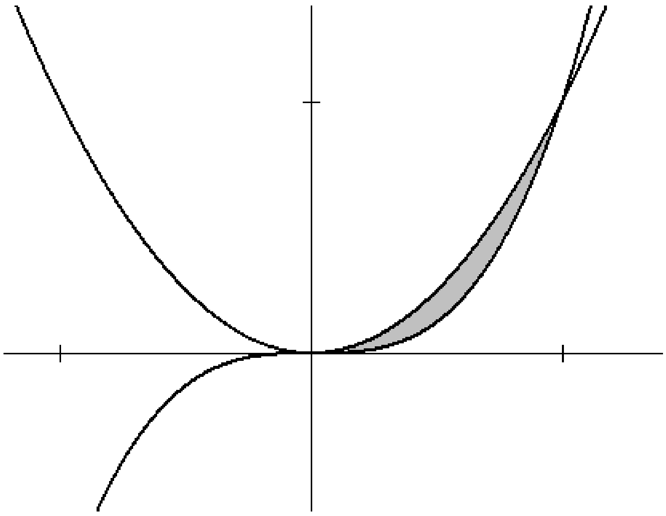
Thus the only intersections of the two curves are at $\underline{x = 0 \text{ and } x = 1}$. Establishing that is worth $\underline{\text{2 points}}$.

So the area we're looking for is $\left|\int_0^1 [f(x) - g(x)]\, dx\right| = \underline{\left|\int_0^1 (x^2 - x^3) dx\right|}$; getting that far is worth 3 points. Let us compute

$$\int_0^1 (x^2 - x^3)dx \; = \; \underline{\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1} \qquad \text{(worth } \underline{4 \text{ points}}\text{)}$$

$$= \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}} \quad \text{or} \quad \boxed{0.08333\cdots} \; .$$

This problem does not actually require a graph, but some students might understand it better if they have one. See below. I have shaded the region that we're finding the area of.



This is a problem on which a graphing calculator, if used inexpertly, could lead you astray. The region whose area you're finding is rather thin, and it might not be visible at all if your calculator's view is zoomed out too much. Several students incorrectly concluded that there was **no** completely enclosed region, and thus they gave the answer of "infinity" or "zero" or "does not exist." I gave those students zero credit. They should have calculated that the two curves have *two* points of intersection — i.e., that the equation $x^2 = x^3$ has *two* solutions. One of the things that I stressed in class was that finding the intersections of the curves was an important component of problems of this type.

A few students came up with an answer of $-1/12$, evidently by making the wrong guess about which curve was higher and which curve was lower. A bit of common sense should have told them that areas are always supposed to be positive numbers. I deducted one point for that error.

(5 points) $\displaystyle\int_2^\infty \frac{1}{(x+1)^2}dx$

Solution. This is page 518 problem 22. Making the substitutions

$$u = x+1, \qquad du = dx$$

is worth 1 point. Then

$$\int_{x=2}^{x=\infty} \frac{1}{(x+1)^2}dx \;=\; \int_{u=3}^{u=\infty} \frac{1}{u^2}du \quad \text{(worth 2 points)}$$

$$= \lim_{R\to\infty} \int_3^R u^{-2}du \;=\; \lim_{R\to\infty} \left[-u^{-1}\right]_3^R \quad \text{(worth 3 points)}$$

$$= \lim_{R\to\infty} \left[\frac{-1}{R} + \frac{1}{3}\right] \quad \text{(worth 4 points)} \;=\; \boxed{\frac{1}{3}}.$$

Alternatively, without substitutions,

$$\int_2^\infty \frac{1}{(x+1)^2}\,dx \;=\; \lim_{R\to\infty} \int_2^R (x+1)^{-2}dx \quad \text{(worth 2 points)}$$

$$= \lim_{R\to\infty} \left[-(x+1)^{-1}\right]_2^R \quad \text{(worth 3 points)}$$

$$= \lim_{R\to\infty} \left[\frac{-1}{R} + \frac{1}{3}\right] \quad \text{(worth 4 points)} \;=\; \boxed{\frac{1}{3}}.$$

An answer of $-1/3$ was worth 4 points, but I was disappointed — the integrand is positive; shouldn't the integral be too?

Some other common erroneous answers were $1/2$ and $1/4$, resulting from errors in the use of the substitution $u = x+1$; I gave 4 points for either of those answers if that appeared to be its source.

Adding a "$+\,C$" to the end of your answer cost a 1-point penalty. It doesn't belong there. Indefinite integrals should end in a "$+C$" (infinitely many answers, all differing from one another by constants) but for definite integrals (only one answer) you have $C - C$ which cancels out.

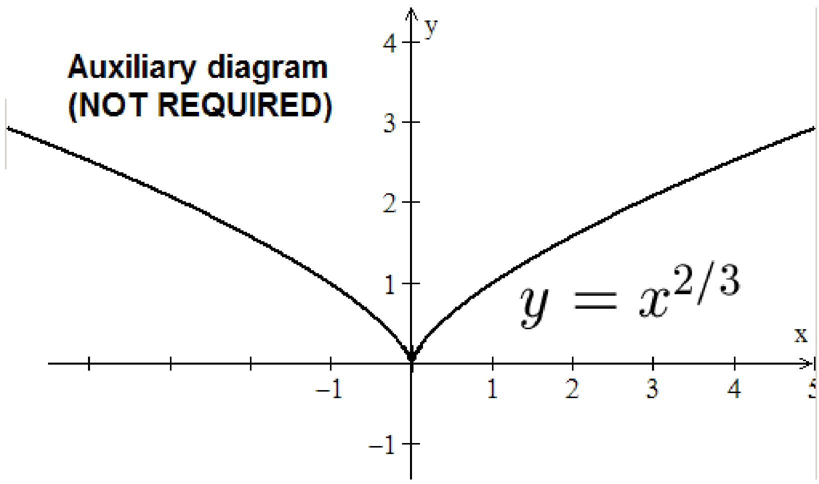
(5 points) Sketch the graph of the function

$$h(x) = (x-1)^{2/3} + 1.$$

If it has any of these features, label them in your graph: x-intercepts, y-intercept, horizontal and/or vertical asymptotes, relative maxima and minima, inflection points. Also, a glance at your sketch should make it evident where the function is defined, and where it is increasing, decreasing, concave upward, or concave downward.

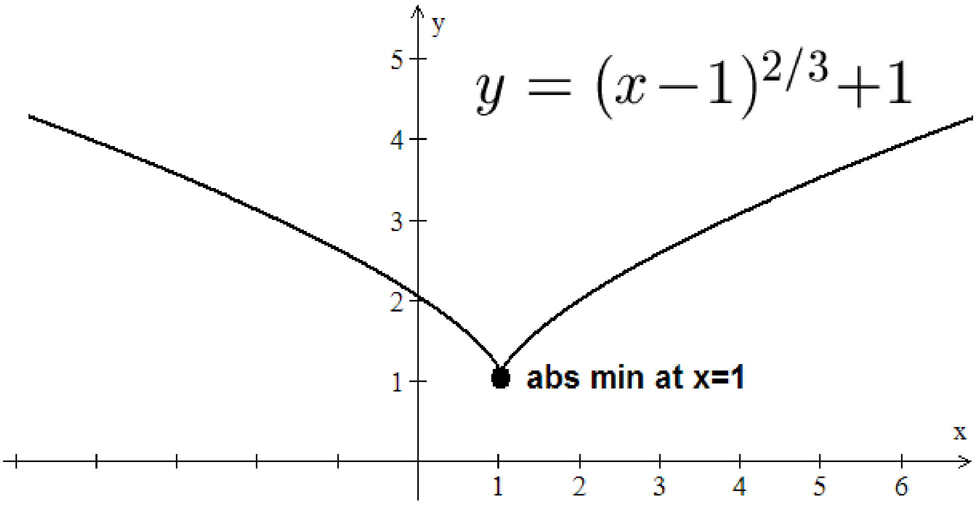
Solution. This is page 293 problem 59. You could get this curve by starting with the picture for $y = x^{2/3}$ (if you know that one), and shifting it one unit to the right and one unit up.

So let's describe $y = x^{2/3}$ first. It is defined everywhere and nonnegative everywhere; it has an absolute minimum at $(0, 0)$. Its derivatives are $y' = \frac{2}{3}x^{-1/3}$, which is positive or negative when x is, except that it is undefined at $x = 0$; and $y'' = -\frac{2}{9}x^{-4/3}$, which is negative everywhere except that it is undefined at $x = 0$. From these results we conclude that the curve for $y = x^{2/3}$ is decreasing on $(-\infty, 0)$ and increasing on $(0, +\infty)$, and concave downward on both those intervals.



Similarly, $y = (x-1)^{2/3}+1$ has an absolute minimum at $(1, 1)$ (and so it has no x-intercept). It is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$, and concave downward on both those intervals. Its y-intercept is at $(0, 2)$, since $h(0) = 2$. There are no asymptotes.

Some additional information (optional): $\left(x^{2/3}\right)'$ blows up when x gets near 0, and similarly $h'(x)$ blows up when x gets near 1, so $h(x)$ has a vertical tangent at $x = 1$.



For full credit, the graph must show the curves in the right shape, with minimum and intercept at the right locations. Showing the graph as two straight line segments in a "vee" shape does not show the concavity, and costs a point. Likewise, if the graph dips downward at the outer edges, to make an M shape, that's wrong.

The point $(1, 1)$ is both an absolute minimum and a relative minimum. Mentioning both of those, or either one of thosse, or just that it's a "min," would be enough.

The following should NOT be present in this graph, and should not be mentioned in your writings about the graph. I did not require you to mention that they were present, but I deducted a point if you asserted that any of them were present: Horizontal asymptote, vertical asymptote, inflection point, relative or absolute maximum, or any interval on which the curve is concave upward.

(6 points) On the interval $-2 \leq x \leq 1$, the function $f(x) = \dfrac{1}{x^2 + 2x + 5}$ has

$$\begin{pmatrix} \text{absolute} \\ \text{maximum} \\ \text{value} \end{pmatrix} = \boxed{} \qquad \begin{pmatrix} \text{absolute} \\ \text{minimum} \\ \text{value} \end{pmatrix} = \boxed{}.$$

Solution. This is page 306 problem 36. We need to check the values of $f(x)$ at all the places where $f'(x)$ doesn't exist or $f'(x)=0$, as well as the endpoints of the interval. From $f(x)=(x^2+2x+5)^{-1}$, let us compute

$$\begin{aligned} f'(x) &= -(x^2+2x+5)^{-2}(x^2+2x+5)' \\ &= -(x^2+2x+5)^{-2}(2x+2) \end{aligned}$$

Computing $\underline{f'(x)}$ correctly is worth <u>1 point</u>.

Now observe that f' vanishes only at $x=-1$, so the only places we need to check are at $\underline{x=-2,-1,1}$; identifying those three numbers gets us up to <u>4 points</u>. Now compute

$$f(-2)=\frac{1}{4-4+5}=\frac{1}{5},\qquad f(-1)=\frac{1}{1-2+5}=\frac{1}{4},\qquad f(1)=\frac{1}{1+2+5}=\frac{1}{8}.$$

Among those numbers, 1/4 is the highest and 1/8 is the lowest. Thus

Absolute maximum is 1/4 (or 0.25); absolute minimum is 1/8 (or 0.125).

I also gave full credit for answers of -1 and 1, respectively, since those are the x-values where the max and min occur. I gave 4 points if the correct answers were given but in reversed order. I gave 3 points if one correct answer was given and there wasn't much work showing any understanding of how to get the other answer.

(5 points) Find the derivative of $y=x^{x+2}$. *Hint:* Although you may use any method you like, I think this problem is probably easiest if you use the method of logarithmic differentiation.

Solution. This is problem 48 on page 377. Begin by noting that

$$\ln y = (x+2)\ln x \qquad \text{(worth 2 points).}$$

Differentiate both sides of that equation; we obtain

$$\begin{aligned} \frac{y'}{y} &= (x+2)'(\ln x)+(x+2)(\ln x) \\ &= 1\cdot\ln x+(x+2)\cdot\tfrac{1}{x} \\ &= \left(\ln x+\tfrac{x+2}{x}\right). \end{aligned}$$

Finding that last expression was worth 3 points; finding it and labeling it as teh value of y'/y was worth 4 points.

Now multiply both sides of that equation by y. That yields

$$y' = \boxed{\left(\frac{x+2}{x} + \ln x\right) x^{x+2}} = \boxed{\left(1 + \frac{2}{x} + \ln x\right) x^{x+2}} = \boxed{(x + 2 + x\ln x)\, x^{x+1}}.$$

(5 points) $\displaystyle\int_0^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx = \boxed{}$

Solution. This is page 488 problem 20. Use the substitution

$$u = x^{1/2}, \qquad du = \frac{1}{2}x^{-1/2}dx, \qquad 2du = x^{-1/2}dx,$$

x	u
4	2
0	0

Writing that much correctly is worth 3 points. Now

$$\int_0^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int_{x=0}^{x=4} e^{\sqrt{x}} \cdot x^{-1/2}dx = \int_{u=0}^{u=2} e^u \cdot 2du$$

(now we're up to 4 points)

$$= [2e^u]_0^2 = \boxed{2e^2 - 2} = \boxed{12.778\cdots}.$$

(5 points) (Circle your answer.) $\int \sqrt{x}\, \ln x\, dx =$

Solution. This is page 488 problem 16. We will use integration by parts,

$$(IP) \qquad \int u'v dx = uv - \int uv' dx$$

with

$$u' = \sqrt{x}, \qquad v = \ln x, \qquad u = \frac{2}{3}x^{3/2}, \qquad v' = x^{-1}.$$

Thus

$$\begin{aligned} & \int \left(x^{1/2}\right)(\ln x)\,dx \\ = \ & \int \left(\tfrac{2}{3}x^{3/2}\right)'(\ln x)\,dx \\ \overset{IP}{=} \ & \left(\tfrac{2}{3}x^{3/2}\right)(\ln x) - \int \left(\tfrac{2}{3}x^{3/2}\right)(\ln x)'\,dx \qquad \text{(worth 3 points)} \\ = \ & \tfrac{2}{3}x^{3/2}\ln x - \tfrac{2}{3}\int x^{3/2}x^{-1}dx \\ = \ & \tfrac{2}{3}x^{3/2}\ln x - \tfrac{2}{3}\int x^{1/2}dx \qquad \text{(worth 4 points)} \\ = \ & \boxed{\frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C} = \boxed{\frac{2}{3}x^{3/2}\left(\ln x - \frac{2}{3}\right) + C}. \end{aligned}$$

Common errors:

Omitting the $+C$ on the end cost 1 point.

Some students apparently calculated $(\ln x)\left(\int \sqrt{x}dx\right) = x^{3/2}\ln x + C$, a complete lack of understanding. Zero credit for that answer.

Total number of points is 50.