Math 140 Test 3, Fri 6 Nov 2009, 4 pages, 35 points, 50 minutes. ANSWERS

(7 points)

white version	orange version
If $f(x) = x^3 + 3x^{1/3}$ on the interval $-1 \leq x \leq 1$, then the highest value taken by this function on that interval is $f(x) = \boxed{}$.	If $g(x) = x^4 + 2x^3 + x^2 + 7$ on the interval $-5 \leq x \leq 5$, then the lowest value taken by this function on that interval is $g(x) = \boxed{}$.

Solutions

Compute $f'(x) = 3x^2 + x^{-2/3}$ — that much is worth 3 points. Then $f'(x)$ is undefined at $x = 0$ (worth 1 pont) and positive everywhere else. Thus the only values we need to consider are $x = 0$ and the endpoints of the interval. Compute f (not f') at those values: $f(-1) = -4$, $f(0) = 0$, $f(1) = 4$. (That much is worth 2 points.) Among those three numbers, $\boxed{4}$ is highest.	Compute $g'(x) = 4x^3 + 6x^2 + 2x$; that much is worth 2 points. Factor it as $g'(x) = 2x(x+\frac{1}{2})(x+1)$ (1 point) to identify that its roots are $0, -1, -1/2$ (another point). Each of those appears as a single factor, and thus is the location of a sign change in g' — that is, we have g' negative on $(-\infty, -1)$ and $(-1/2, 0)$, and positive on $(-1, -1/2)$ and $(0, \infty)$. Therefore g has hence g has relative minima at $x = -1$ and $x = 0$, and a relative maximum at $x = -1/2$. Since we're only looking for minima, we don't need to bother computing the value of $g(-1/2)$ (but noticing that is optional). Now compute $g(-1) = 7$ and $g(0) = 7$ (worth 2 points), so the minimum is $\boxed{7}$.

(5 points)

white version	orange version
If $\dfrac{5}{3^{-2x}-2} = 7$, then $x =$ [ ]	If $3^{2t+1} = \dfrac{1}{9^{t-1}}$, then $t =$ [ ]

Solutions

$\dfrac{5}{3^{-2x}-2} = 7$	$3^{2t+1} = \dfrac{1}{9^{t-1}}$
$3^{-2x} - 2 = \dfrac{5}{7}$	$3^{2t+1} = (3^{-2})^{t-1}$
$3^{-2x} = \dfrac{5}{7} + 2 = \dfrac{19}{7}$	$3^{2t+1} = 3^{-2(t-1)}$
$3^{2x} = \dfrac{7}{19}$	$2t + 1 = -2(t-1)$
$2x = \log_3 \dfrac{7}{19}$	$2t + 1 = -2t + 2$
$x = \boxed{\dfrac{1}{2}\log_3\left(\dfrac{7}{19}\right)}$ or $\boxed{-0.45}$	$t = \boxed{\dfrac{1}{4}} = \boxed{0.25}$

Generally, one point was deducted for each arithmetic/algebra mistake, 2 points for mild conceptual errors, 3 points for severe conceptual errors, more for total cluelessness. One common error was in the proper use of the law of distribution of multiplication over addition — for instance, it is correct to say $7(3^{-2x} - 2) = 7 \cdot 3^{-2x} - 14$, but $7 \cdot 3^{-2x}$ is *not* equal to 3^{-14x} or several other things.

(5 points)

white version	orange version
If $f(x) = e^{1/x}$, then $f'(x) =$ [ ] .	If $g(x) = e^{x^2 + 2x}$, then $g'(x) =$ [ ] .

Solution. In both cases, we use the formula $\frac{d}{dx}e^{p(x)} = p'(x)e^{p(x)}$.

Use $p(x) = x^{-1}$, so $p'(x) = -x^{-2}$, and the answer is $f'(x) = \boxed{-x^{-2}e^{1/x}}$.	Use $p(x) = x^2 + 2x$, so $p'(x) = 2x + 2$, and $g'(x) = \boxed{(2x+2)e^{x^2+2x}}$.

(9 points) Same problem on both tests:

An open box (i.e., bottom and four sides but no top) with a square base is to be made with a total volume of 32 cubic feet. All five rectangles (bottom and four sides) are to be made of the same material, so to minimize the cost, we must minimize the total of their areas. What is the smallest possible total area?

Solution. Let s be the length of one side of the base. Then the base has area s^2. Since the volume is 32, the height of the box must be $32s^{-2}$. Then the four side rectangles each have area $32s^{-1}$, so the total area is $A(s) = s^2 + 4 \cdot 32s^{-1} = s^2 + 128s^{-1}$. That function blows up as $s \to 0$ and as $s \to \infty$ — i.e., it is largest at those values — so it is smallest at some intermediate value. Compute $A'(s) = 2s - 128s^{-2} = 2s^{-2}(s^3 - 64) = 2s^{-2}(s^3 - 4^3)$, which vanishes at $s = 4$. Thus the optimum is achieved at $s = 4$, and then we have $A(4) = 4^2 + 128 \cdot 4^{-1} = 16 + 32 = \boxed{\text{48 square feet}}$.

(9 points) (White version.) Sketch the graph of $y = \frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{15}{2}x^2$

Solution. We have $f(0) = 0$, so the y-intercept is at $(0,0)$. What about x-intercepts? Set $y = 0$. After factoring out x^2, we are left with $\frac{1}{4}x^2 - \frac{2}{3}x - \frac{15}{2} = \frac{1}{12}\left(3x^3 - 8x - 90\right)$, which has roots at

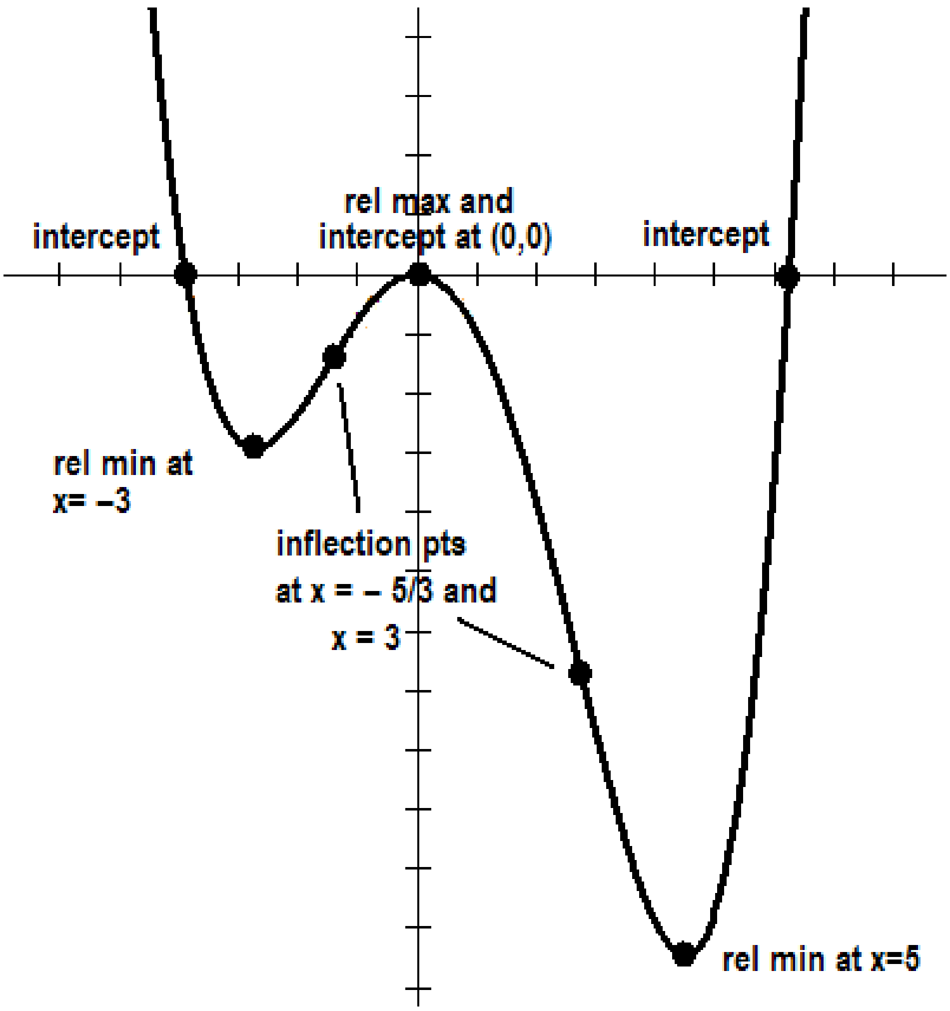
$$x = \frac{8 \pm \sqrt{8^2 + 4 \cdot 3 \cdot 90}}{2 \cdot 3} = \frac{8 \pm 2\sqrt{4^2 + 270}}{6} = \frac{8 \pm 2\sqrt{286}}{6} = \frac{4 \pm \sqrt{286}}{3}.$$

Next, $y' = x^3 - 2x^2 - 15x = x(x-5)(x+3)$ with roots $x = 0, 5, -3$.

And then $y'' = 3x^2 - 4x - 15$, which has roots at

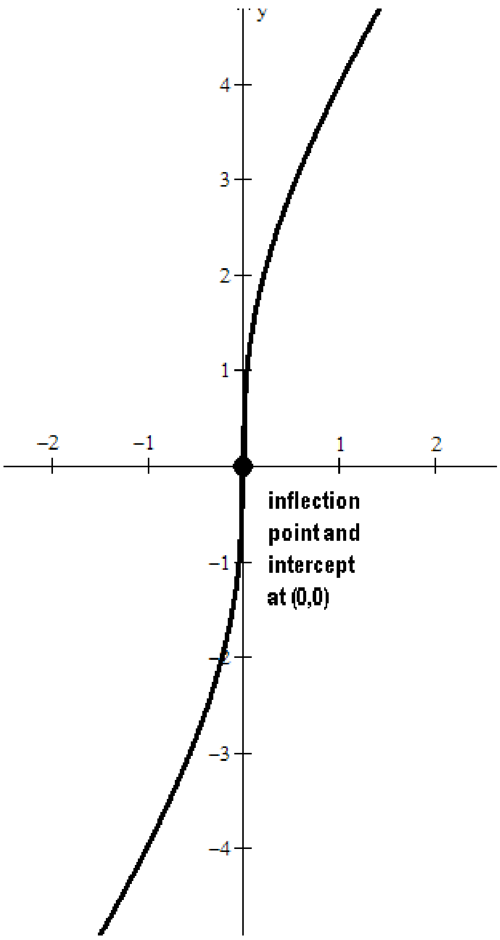
$$x = \frac{4 \pm \sqrt{4^2 + 4 \cdot 3 \cdot 15}}{2 \cdot 3} = \frac{4 \pm 2\sqrt{4 + 45}}{6} = \frac{4 \pm 14}{6} = \frac{2 \pm 7}{3} = \{3, \ -5/3\}.$$

x		-3		$-5/3$		0		3		5	
y'	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$
y''	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$
shape	$\lfloor$	min	$\rfloor$	inflec	$\lceil$	max	$\rceil$	inflec	$\lfloor$	min	$\rfloor$



(Orange version) Sketch the graph of $y = x + 3x^{1/3}$

Solution. $y' = 1 + x^{-2/3}$, which is positive everywhere, except that it is undefined at $x = 0$. Then $y'' = -\frac{2}{3}x^{-5/3}$, which is positive for $x < 0$, negative for $x > 0$, and undefined at $x = 0$. Thus the function y is increasing everywhere, with a vertical tangent at $x = 0$. It is concave upward for $x < 0$, inflection point at $x = 0$, concave downward for $x > 0$. The only intercept is at $(0, 0)$. There are no asymptotes.



Grading on the graphing problems (either version) was extremely lenient, because I forgot to give detailed instructions. I had intended to say "show and label all intercepts, relative maxima, relative minima, inflection points, and other features of interest in the graph." But I took it for granted that anyone would understand that to be part of the instructions, so I neglected to mention it — and unfortunately it was not understood, but I could not penalize students for not following instructions that I did not give.