Math 140 Test 2, Mon 12 Oct 2009, 3 pg, 35 pts, 50 min. **ANSWERS**

(5 points)

Yellow version: Using "the method of differentials" (i.e., using a derivative in the appropriate fashion), find the approximate value of $\sqrt{100.1}$. (You are *not* asked to find the *exact* value of $\sqrt{100.1}$.)

White version: Using "the method of differentials" (i.e., using a derivative in the appropriate fashion), find the approximate value of $\sqrt{99.9}$. (You are *not* asked to find the *exact* value of $\sqrt{99.9}$.)

Solutions. In both cases, we use $f(x) = x^{1/2}$ and $f'(x) = \frac{1}{2}x^{-1/2}$. Plugging in $x = 100$ yields $f(x) = 10$ and $f'(x) = \frac{1}{2} \cdot \frac{1}{10} = 0.05$. What we're looking for is $f(x+h) \approx f(x) + hf'(x)$; that is, $f(100+h) \approx 10 + 0.05h$.

When $h = 0.1$, we get $\sqrt{100.1} \approx 10 + (0.05)(0.1) = \boxed{10.005}$.
(In case you're wondering, the exact value is $\sqrt{100.1} = 10.00499875\cdots$.)

When $h = -0.1$, we geet $\sqrt{99} \approx 10 - (0.05)(0.1) = \boxed{9.995}$.
(In case you're wondering, the exact value is $\sqrt{99.9} = 9.9949987\cdots$.)

Common errors, worth 1 point each:
Knowing the formula $f(x+h) \approx f(x) + hf'(x)$
Figuring out what x is
Figuring out what $f(x)$ is
Figuring out what $f'(x)$ is
Figuring out what h is
Carrying out the computation of $f(x) + hf'(x)$
Any arithmetic error

(6 points)

Yellow version: Find an equation for the line that is tangent to the curve $x^5y + 2xy^7 = 3$ at the point $(1, 1)$.

White version: Find an equation for the line that is tangent to the curve $x^{1/2}y + 2xy^3 = 3$ at the point $(1, 1)$.

Solution. The equation for a line is of the form $y = mx + b$, where m and b are some constants (i.e., *numbers* – not functions of x). Any answer of that form received at least one point.

An equation for a line that passes through the point $(1, 1)$ is of the form $y - 1 = m(x - 1)$ or $y = mx + (1 - m)$, for some number m. Any answer of that form received at least 2 points.

Finding m was worth the other 4 points. Many students understood the method but made arithmetic errors along the way; they received most of the credit. One particularly common error was to transform $(-2) + (-5)$ (which should be -7) into $(-2)(-5)$ (which is 10, but in some cases was computed as -10); I don't know why that particular error was so common; it cost one point.

How do you find m? The main step is the implicit differentiation. Many students knew how to do that, but many other students were completely clueless about that (and lost all 4 of those points). Indeed, some students even seemed to think that the derivative of pq is $p'q'$ — thus, for instance, they thought that the derivative of $x^{1/2}y$ is $\frac{1}{2}x^{-1/2}y'$. No! The derivative of pq actually is $p'q + pq'$, and the derivative of $x^{1/2}y$ actually is $\frac{1}{2}x^{-1/2}y + x^{1/2}y'$.

Yellow version	White version
$x^5y + 2xy^7 = 3$	$x^{1/2}y + 2xy^3 = 3$

Implicitly differentiate both sides with respect to x:

Yellow version	White version
$5x^4y + x^5y' + 2y^7 + 14xy^6y' = 0$	$\frac{1}{2}x^{-1/2}y + x^{1/2}y' + 2y^3 + 6xy^2y' = 0$

(Getting that far was worth 2 points.) (Some students next solved for y', but that's doing it the hard way.) Next, plug in $x = y = 1$:

Yellow version	White version
$5 + y' + 2 + 14y' = 0$	$\frac{1}{2} + y' + 2 + 6y' = 0$

(Getting that far was worth 3 points.) Next, simplify; solve for y'

Yellow version	White version
$7 + 15y' = 0$	$\frac{5}{2} + 7y' = 0$
$y' = -7/15$	$y' = -5/14$

That's m, worth 4 points. Use it as the slope of a line through $(1, 1)$:

Yellow version	White version
$\boxed{y - 1 = (-7/15)(x - 1)}$	$\boxed{y - 1 = (-5/14)(x - 1)}$
or $15(y - 1) + 7(x - 1) = 0$	or $14(y - 1) + 5(x - 1) = 0$
or $\boxed{7x + 15y = 22}$	or $\boxed{5x + 14y = 19}$
or $\boxed{y = -\frac{7}{15}x + \frac{22}{15}}$	or $\boxed{y = -\frac{5}{14}x + \frac{19}{14}}$

Some students, before plugging in $x = y = 1$, solved for y':

Yellow version	White version
$y = -\dfrac{5x^4 + 2x^7}{x^5 + 14xy^6}$	$y = -\dfrac{\frac{1}{2}x^{-1/2}y + 2y^3}{x^{1/2} + 6xy^2}$

— an unnecessary complication, but I gave 3 points for getting that far in finding m.

Page 2:

(6 points)

Yellow version: At 2 pm, Jack is driving west from Nashville at 70 miles per hour, and he is 30 miles west of Nashville. Sue is driving south from Nashville at 50 miles per hour, and she is 20 miles south of Nashville.

What is the distance between them? $\boxed{}$

At what rate is the distance between them increasing? $\boxed{}$

White version: At 9 pm, Elaine is driving east from Nashville at 60 miles per hour, and she is 20 miles east of Nashville. David is driving north from Nashville at 70 miles per hour, and he is 30 miles north of Nashville.

What is the distance between them? $\boxed{}$

At what rate is the distance between them increasing? $\boxed{}$

Solution. We use the fact that $z^2 = x^2 + y^2$, so $2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}$.

.	Yellow version	White version
Let $x =$	Jack's distance from Nashville	Elaine's distance from Nashville
Let $y =$	Sue's distance from Nashville	David's distance from Nashville
Let $z =$	distance between Jack and Sue	distance between David and Elaine
Given data: at the given time,	$dx/dt = 70, \quad x = 30$ $dy/dt = 50, \quad y = 20$	$dx/dt = 60, \quad x = 20$ $dy/dt = 70, \quad y = 30$

$$z = \sqrt{x^2 + y^2} = \sqrt{20^2 + 30^2} = 10\sqrt{2^2 + 3^2} = \boxed{10\sqrt{13} \text{ mi}} \approx \boxed{36.06 \text{ mi}}$$

	Yellow version	White version
$\dfrac{dz}{dt} = \dfrac{x(dx/dt) + y(dy/dt)}{z} =$	$\dfrac{30 \cdot 70 + 20 \cdot 50}{10\sqrt{13}}$	$\dfrac{20 \cdot 60 + 30 \cdot 70}{10\sqrt{13}}$
	$= \boxed{\dfrac{310}{\sqrt{13}} \text{ mph}}$	$= \boxed{\dfrac{330}{\sqrt{13}} \text{ mph}}$
	$= \boxed{85.98 \text{ mph}}$	$= \boxed{91.53 \text{ mph}}$

(6 points)

Yellow version: Find the intervals where the function $f(x) = \frac{1}{2}x^2 - 2x^{1/2}$ is increasing or decreasing; find its relative maxima and relative minima, if any. Circle your answers.

White version: Find the intervals where the function $g(x) = 2x^3 - 3x^2 + 5$ is increasing or decreasing; find its relative maxima and relative minima, if any. Circle your answers.

Solution.

$$f'(x) = x - x^{-1/2} = x^{-1/2}(x^{3/2} - 1) = \begin{cases} \text{undefined for } x \leq 0 \\ \text{negative for } 0 < x \leq 1 \\ 0 \text{ for } x = 1 \\ \text{positive for } x > 1 \end{cases}$$ Hence

$$\boxed{\begin{array}{c}\text{The function } f \text{ is decreasing on } (0,1), \\ \text{increasing on } (1,\infty), \text{ and has a} \\ \text{relative minimum at } x = 1.\end{array}}$$

(Do *not* deduct any points if the student says the function also has a relative maximum at $x = 0$. However, if you use the definition our textbook uses, that is not true;. Our textbook only permits a "relative maximum" or "relative minimum" to occur at a point in an *open* interval in the domain of the function – and hence not at an endpoint of the domain.)

$g'(x) = 6x^2 - 6x = 6x(x-1)$ is positive on $(-\infty$,0) and on $(1,\infty)$, and negative on $(0,1)$. Hence

$$\boxed{\begin{array}{c}\text{The function } g \text{ is decreasing on } (0,1), \\ \text{increasing on } (-\infty,0) \text{ and } (1,\infty), \text{ and has} \\ \text{a relative maximum at } x = 0 \\ \text{and relative minimum at } x = 1.\end{array}}$$

Page 3:

(6 points)

Yellow version: Find the derivative of $p(x) = \left[1 + \left(x^3 + 1\right)^{1/2}\right]^{100}$.

White version: Find the derivative of $q(x) = \left[1 + \left(x^2 + 1\right)^{1/3}\right]^{100}$.

Solution. In both cases we will use $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{du} \cdot \frac{du}{dx}$ with

$$\begin{aligned} z &= y^{100}, & \tfrac{dz}{dy} &= 100y^{99} \\ y &= 1+u^b, & \tfrac{dy}{du} &= bu^{b-1} \\ u &= x^a+1, & \tfrac{du}{dx} &= ax^{a-1}. \end{aligned}$$

Thus

$$\frac{dz}{dx} = 100aby^{99}u^{b-1}x^{a-1} = 100abx^{a-1}(x^a+1)^{b-1}\left[1+(x^a+1)^b\right]^{99}.$$

Yellow version: Taking $a=3$ and $b=1/2$ yields

$p'(x) = \boxed{150x^2(x^3+1)^{-1/2}\left[1+(x^3+1)^{1/2}\right]^{99}}$ or $\boxed{\dfrac{150x^2\left[1+\sqrt{x^3+1}\,\right]^{99}}{\sqrt{x^3+1}}}$. Other forms of the answer will also be accepted if they are not significantly more complicated in appearance.

White version: Taking $a=2$ and $b=1/3$ yields

$q'(x) = \boxed{\tfrac{200}{3}x(x^2+1)^{-2/3}\left[1+(x^2+1)^{1/3}\right]^{99}}$ or $\boxed{\dfrac{200x\left[1+\sqrt[3]{x^2+1}\right]^{99}}{3(x^2+1)^{2/3}}}$. Other forms of the answer will also be accepted if they are not significantly more complicated in appearance.

(6 points)

Yellow version: Find the third derivative of $f(x) = \dfrac{3x}{x+1}$

White version: Find the third derivative of $g(x) = \dfrac{2x}{x-1}$

Solution. Many students did not follow the principle that if you simplify an expression as early as you can, then subsequent computations with it will be made less troublesome. Thus, you should simplify the first derivative before computing the second derivative. Many students didn't do that, and this led to all sorts of errors.

After you get an expression of the form $(x+k)^p$ (where k and p are constants), you no longer need to use the quotient rule; that saves a lot of work.

$$\text{Yellow version:}\quad f'(x) = \left(\frac{3x}{x+1}\right)' = \frac{(3x)'(x+1)-(3x)(x+1)'}{(x+1)^2}$$
$$= \frac{3(x+1)-(3x)}{(x+1)^2} = \frac{3}{(x+1)^2} = 3(x+1)^{-2},$$

or, more easily,

$$f(x) = \frac{3x}{x+1} = \frac{3(x+1)-3}{x+1} = 3-3(x+1)^{-1}, \quad \text{so} \;\; f'(x) = 3(x+1)^{-2}.$$

After that, $f''(x) = -6(x+1)^{-3}$ and $f'''(x) = \boxed{18(x+1)^{-4}}$.

$$\text{White version:}\quad g'(x) = \left(\frac{2x}{x-1}\right)' = \frac{(2x)'(x-1)-(2x)(x-1)'}{(x-1)^2}$$
$$= \frac{2(x-1)-(2x)}{(x-1)^2} = \frac{-2}{(x-1)^2} = -2(x-1)^{-2}$$

or, more easily,

$$g(x) = \frac{2x}{x-1} = \frac{2(x-1)+2}{x-1} = 2+2(x-1)^{-1}, \quad \text{so} \;\; g'(x) = -2(x-1)^{-2}.$$

After that, $g''(x) = 4(x-1)^{-3}$ and $g'''(x) = \boxed{-12(x-1)^{-4}}$.

Point breakdown:
1 point for proper use of either product rule or quotient rule
1 point for proper calculation and simplification of the first derivative
1 point for calculation of second derivative
1 point for simplification of second derivative
1 point of calculation of third derivative
1 point for simplification of third derivative

Note: Here is a way to avoid using the quotient rule in calculating the first derivative:

$$f(x) = \frac{3x}{x+1} = \frac{3(x+1)-3}{x+1} = 3-3(x+1)^{-1}, \qquad \text{hence} \quad f'(x) = 3(x+1)^{-2}$$

$$g(x) = \frac{2x}{x-1} = \frac{2(x-1)+2}{x-1} = 2+2(x-1)^{-1}, \qquad \text{hence} \quad g'(x) = -2(x-1)^{-2}$$