Quiz 3, 23 October 2009, 20 minutes, 1 page, 15 points. **ANSWERS.**

(6 points)

| white version | green version |
|---|---|
| Find the absolute maximum value of $f(x)$, where $f(x) = x^4 - 8x^2$, on the interval $-4 \le x \le 4$. Answer: $f(x) =$ $\boxed{\phantom{000}}$ . | Find the absolute minimum value of $g(x)$, where $g(x) = \frac{1}{2}x^2 + x^{-1}$, on the interval $\frac{1}{2} \le x \le 2$. Answer: $g(x) =$ $\boxed{\phantom{000}}$ . |

Solutions

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| $f'(x) = 4x^3 - 16x = 4x(x^2-4) = 4x(x-2)(x+2)$, which vanishes at $x = 0, \pm 2$. Thus we must check the values of $f(x)$ at those three numbers, as well as at the endpoints $\pm 4$. However, we can save a little work: $f''(x) = 12x^2 - 16 = 4(3x^2 - 4)$ is positive at $x = \pm 2$, so $f(x)$ can only have a relative minimum at those locations, not a relative maximum. This leaves us to check $f(0) = 0$, $f(4) = f(-4) = 128$. Among those numbers, $\boxed{128}$ is largest. I will also allow partial credit for the answer $\boxed{4}$ or $\boxed{-4}$ or $\boxed{\pm 4}$, since that is where the relative maximum occurs. | $g'(x) = x - x^{-2} = x^{-2}(x^3 - 1) = x^{-2}(x - 1)(x^2 + x + 1)$, which vanishes at $x = 1$. We need to check that point, and the endpoints. Compute $g(1/2) = 2.125$, $g(1) = 1.5$, and $g(2) = 2.5$. The absolute minimum is $\boxed{1.5}$. Partial credit will be give for a value of $\boxed{1}$, the value of $x$ where the minimum occurs. |

(9 points)

| green version | white version |
|---|---|
| A rectangular garden, 120 square feet in area, is to be created by fencing in a portion of a field. For reasons that will not be explained here, the north side of the garden requires a different type of fencing material than the other three sides. The fence on the north side will cost two dollars per foot, and the fence on the other three sides of the rectangle will cost three dollars per foot. | A rectangular garden, 90 square feet in area, is to be created by fencing in a portion of a field. For reasons that will not be explained here, the different sides of the garden require different kinds of fencing material. The fence on the north side will cost three dollars per foot, the fence on the south will cost two dollars per foot, the fence on the east and west sides will cost one dollar per foot. |

Find the smallest possible cost of the fence:

$$\begin{pmatrix} \text{total} \\ \text{cost of} \\ \text{fence} \end{pmatrix} = \boxed{\phantom{000}}$$

and the dimensions of the rectangle yielding that price.

$$\begin{pmatrix} \text{length} \\ \text{of east} \\ \text{or west} \\ \text{side} \end{pmatrix} = \boxed{\phantom{000}} \qquad \begin{pmatrix} \text{length} \\ \text{of north} \\ \text{or south} \\ \text{side} \end{pmatrix} = \boxed{\phantom{000}}$$

*Solutions.* Let $n, e, w, s$ be the cost, per foot, of fencing for the north, east, west, and south sides of the rectangle; and let $N, E, W, S$ be the lengths of those fences. The the cost of the north fence is $nN$, and similarly for each of the other sides. Thus the total cost is $C =$

$nN + eE + wW + sS$.

| green version | white version |
|---|---|
| $n = 2$ | $n = 3$ |
| $e = 3$ | $e = 1$ |
| $w = 3$ | $w = 1$ |
| $s = 3$ | $s = 2$ |
| $C = 2N + 3E + 3W + 3S$ | $C = 3N + E + W + 2S$ |

However, it's a rectangle we're dealing with here, so $N = S$ and $E = W$. Thus the total cost is $C = (n + s)N + (e + w)E$.

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| $C = 5N + 6E$ | $C = 5N + 2E$ |

That's the cost function, as a function of two variables; that equation is worth 1 point.

We're given the total area, $a = NE$. (Recognizing that was worth 1 point.)

| | |
|---|---|
| $NE = 120$ | $NE = 90$ |

We can eliminate either one of the variables — either $N$ or $E$ — and then solve the problem in terms of the other variable.

If we eliminate $E$, we compute $E = aN^{-1}$, and so

$$C(N) = (n + s)N + (e + w)aN^{-1},$$

where $N$ is a variable and $a, n, s, e, w$ are given constants. That's the cost function as a function of one variable; that's worth 2 points. Differentiating it correctly is worth another point:

$$C'(N) = (n + s) - (e + w)aN^{-2}.$$

Thus, in the two versions:

| | |
|---|---|
| $E = 120N^{-1}$ | $E = 90N^{-1}$ |
| $C(N) = 5N + 720N^{-1}$ | $C(N) = 5N + 180N^{-1}$ |
| $C'(N) = 5 - 720N^{-2}$ | $C'(N) = 5 - 180N^{-2}$ |

Finding the critical point(s) is worth another point: The only place where $C'(N)$ is zero is where $n+s = (e+w)aN^{-2}$, or $N = \sqrt{\dfrac{(e+w)a}{n+s}}$.

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| $\boxed{N = 12}$ | $\boxed{N = 6}$ |

Finishing the computation was worth the last 3 points: $E = aN^{-1}$, which yields $E = \sqrt{\dfrac{(n+s)a}{e+w}}$ and $C = 2\sqrt{(n+s)(e+s)a}$.

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|---|---|
| $\boxed{E = 10}$ | $\boxed{E = 15}$ |
| $\boxed{C = 120}$ | $\boxed{C = 60}$ |

<u>If we eliminate $N$</u>, we compute $N = aE^{-1}$, and so

$$C(E) = (n+s)aE^{-1} + (e+w)E,$$

where $E$ is a variable and $a, n, s, e, w$ are given constants. That's the cost function as a function of one variable; it's worth 2 points. Differentiating it correctly is worth another point:

$$C'(E) = -(n+s)aE^{-2} + (e+w).$$

Thus, in the two versions:

| | |
|---|---|
| $N = 120E^{-1}$ | $N = 90E^{-1}$ |
| $C(E) = 600E^{-1} + 6E$ | $C(E) = 450E^{-1} + 2E$ |
| $C'(E) = -600E^{-2} + 6$ | $C'(E) = -450E^{-2} + 2$ |

Finding the critical point(s) is worth another point: The only place where $C'(E)$ is zero is where $(n+s)aE^{-2} = e+w$, or $E = \sqrt{\dfrac{(n+s)a}{e+w}}$.

| | |
|---|---|
| $\boxed{E = 10}$ | $\boxed{E = 15}$ |

Finishing the computation was worth the last 3 points: $N = aE^{-1}$, which yields $N = \sqrt{\dfrac{(e+w)a}{n+s}}$, and $C = 2\sqrt{(n+s)(e+s)a}$.

| | |
|---|---|
| $\boxed{N = 12}$ | $\boxed{N = 6}$ |
| $\boxed{C = 120}$ | $\boxed{C = 60}$ |