More Topics About Power Series

(I don't know why these topics are omitted from our textbook's introduction to power series. These topics ought to be included in any introduction to the subject.)

Finding the Coefficients in a Power Series

If we're given some function $f(x)$ that we know a lot about (such as $\sin x$ or e^x), how can we find its power series? Here is the basic idea. Assume that $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$, where the c_j's are numbers that we're trying to find. Take derivatives, term by term. Thus we compute:

$$\begin{array}{rclcccccccc}
f(x) & = & c_0 & +c_1 x & +c_2 x^2 & +c_3 x^3 & +c_4 x^4 & +c_5 x^5 & +c_6 x^6 & + \cdots \\
f'(x) & = & & c_1 & +2c_2 x & +3c_3 x^2 & +4c_4 x^3 & +5c_5 x^4 & +6c_6 x^5 & + \cdots \\
f''(x) & = & & & 2c_2 & +2\cdot 3c_3 x & +3\cdot 4c_4 x^2 & +4\cdot 5c_5 x^3 & +5\cdot 6c_6 x^4 & + \cdots \\
f'''(x) & = & & & & 2\cdot 3c_3 & +2\cdot 3\cdot 4c_4 x & +3\cdot 4\cdot 5c_5 x^2 & +4\cdot 5\cdot 6c_6 x^3 & + \cdots \\
f^{iv}(x) & = & & & & & 2\cdot 3\cdot 4c_4 & +2\cdot 3\cdot 4\cdot 5c_5 x & +3\cdot 4\cdot 5\cdot 6c_6 x^2 & + \cdots \\
f^{v}(x) & = & & & & & & 2\cdot 3\cdot 4\cdot 5c_5 & +2\cdot 3\cdot 4\cdot 5\cdot 6c_6 x & + \cdots
\end{array}$$

and so on. Now plug in $x = 0$, and all the terms involving x^j for $j > 0$ disappear. That leaves

$$\begin{array}{rclcrcl}
f(0) & = & c_0 + 0 + 0 + 0 + 0 + 0 + \cdots, & \text{hence} & c_0 & = & f(0) \\
f'(0) & = & c_1 + 0 + 0 + 0 + 0 + 0 + \cdots, & \text{hence} & c_1 & = & f'(0) \\
f''(0) & = & 2c_2 + 0 + 0 + 0 + 0 + \cdots, & \text{hence} & c_2 & = & f''(0)/2 \\
f'''(0) & = & 2\cdot 3c_3 + 0 + 0 + 0 + \cdots, & \text{hence} & c_3 & = & f'''(0)/3! \\
f^{iv}(0) & = & 2\cdot 3\cdot 4c_4 + 0 + 0 + \cdots, & \text{hence} & c_4 & = & f^{iv}(0)/4! \\
f^{v}(0) & = & 2\cdot 3\cdot 4\cdot 5c_5 + 0 + \cdots, & \text{hence} & c_5 & = & f^{(5)}(0)/5!
\end{array}$$

and, in general, $\boxed{c_n = f^{(n)}(0)/n!}$.

Example: To find the power series for $\sin x$, calculate

$$\begin{array}{rclrclrcl}
f(x) & = & \sin x, & f(0) & = & 0, & c_0 & = & 0 \\
f'(x) & = & \cos x, & f'(0) & = & 1, & c_1 & = & 1 \\
f''(x) & = & -\sin x, & f''(0) & = & 0, & c_2 & = & 0 \\
f'''(x) & = & -\cos x, & f'''(0) & = & -1, & c_3 & = & -1/3! \\
f^{(4)}(x) & = & \sin x, & f^{(4)}(0) & = & 0, & c_4 & = & 0 \\
f^{(5)}(x) & = & \cos x, & f^{(5)}(0) & = & 1, & c_3 & = & 1/5! \\
f^{(6)}(x) & = & -\sin x, & f^{(6)}(0) & = & 0, & c_4 & = & 0 \\
f^{(7)}(x) & = & -\cos x, & f^{(7)}(0) & = & -1, & c_3 & = & -1/7! \\
f^{(8)}(x) & = & \sin x, & f^{(8)}(0) & = & 0, & c_4 & = & 0 \\
f^{(9)}(x) & = & \cos x, & f^{(9)}(0) & = & 1, & c_3 & = & 1/9! \\
f^{(10)}(x) & = & -\sin x, & f^{(10)}(0) & = & 0, & c_4 & = & 0
\end{array}$$

and thus we have the power series

$$\sin x \quad = \quad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots.$$

Exercises. Find the power series for $\cos x$, e^x, e^{3x}, $\sin 7x$.

Another example. Using the method above with $f(x) = (1-x)^{-1}$, show that

$$\frac{1}{1-x} \quad = \quad 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots.$$

This series is important enough to have a name. It is called the *geometric series.* The formula above is only valid when $|x| < 1$; we will discuss it further a few paragraphs from now.

Power series in disguised form. By making substitutions, show that

$$\begin{aligned}
\sin(x^3) &= x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \frac{x^{27}}{9!} - \cdots \\
\frac{1}{\sqrt{x}} \sin \sqrt{x} &= 1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \frac{x^4}{9!} - \cdots \\
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots \\
\frac{1}{1+x^2} &= 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots
\end{aligned}$$

Integrating series term by term. By integrating series for the function $1/(1-x)$ or the function $1/(1+x^2)$, respectively, show that

$$\begin{aligned}
-\ln(1-x) &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \cdots \\
\arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots
\end{aligned}$$

What is the Radius of Convergence?
and
What do we mean by a power series converging?

(After this topic we'll discuss how to *find* the radius of convergence.)

(In most of the discussion below, by a "number" we will mean any complex number; infinity is not one of the numbers allowed. There is one exception: In the definition of R, we allow any "number" in $[0, \infty]$. It must be a real number, it must be greater than or equal to 0, but it could be infinity.)

Let's start with a simpler question. Instead of power series, let's just consider a series of numbers. For instance, we say that

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots \quad = \quad 2.$$

What does that mean? How can we add up infinitely many numbers?

What it means is that we add up just finitely many of the numbers, and see what we get:

$$\begin{aligned}
&1 &&= 1 \\
&1 + \tfrac{1}{2} &&= 1.5 \\
&1 + \tfrac{1}{2} + \tfrac{1}{4} &&= 1.75 \\
&1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} &&= 1.875 \\
&1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} &&= 1.9375 \\
&1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} &&= 1.96875 \\
&1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \tfrac{1}{64} &&= 1.984375
\end{aligned}$$

The numbers in the right hand column are getting closer and closer to 2. So we say that *the series converges to 2*, or that *the sum of the series is 2*.

Some series do not converge. For instance, the partial sums of the series $1-1+1-1+1-1+\cdots$ are the numbers $1, 0, 1, 0, 1, 0, \ldots$. Those are not getting closer to one limit, so we say that the series $1-1+1-1+1-\cdots$ *does not converge*, or that it *diverges*.

The partial sums of the series $1+2+4+8+16+\cdots$ do not converge to any finite number, so this series also *diverges*, in our terminology. (In some contexts this series would instead be described as "converging to infinity," but that alternate description is not useful for our purposes and we will not use it. We will simply say that this series diverges.)

Now let's turn to a power series. For instance, consider

$$f(x) \quad = \quad 1 + x + x^2 + x^3 + x^4 + \cdots.$$

When we plug in a number for x, we get a series of numbers, which might converge or diverge. For instance,

$$\begin{array}{llllllllllll}
\text{plugging in} & x=\frac{1}{2} & \text{yields} & f(\frac{1}{2}) & = & 1 & +\frac{1}{2} & +\frac{1}{4} & +\frac{1}{8} & +\frac{1}{16} & +\cdots & = \; 2 \quad \text{(convergent);} \\
\text{plugging in} & x=-1 & \text{yields} & f(-1) & = & 1 & -1 & +1 & -1 & +1 & -\cdots & \text{which diverges;} \\
\text{plugging in} & x=2 & \text{yields} & f(\frac{1}{2}) & = & 1 & +2 & +4 & +8 & +16 & +\cdots & \text{which diverges.}
\end{array}$$

In fact (though I won't prove it here) it turns out that

$$f(x) \quad = \quad 1 + x + x^2 + x^3 + x^4 + \cdots \quad = \quad \begin{cases} 1/(1-x) & \text{if} \quad |x| < 1, \\ \text{divergent} & \text{if} \quad |x| \geq 1. \end{cases}$$

So we can define a function by

$$f(x) \quad = \quad 1 + x + x^2 + x^3 + x^4 + \cdots \quad = \quad \begin{cases} 1/(1-x) & \text{if} \quad |x| < 1, \\ \text{undefined} & \text{if} \quad |x| \geq 1. \end{cases}$$

We say that this series has radius of convergence equal to 1; that term is explained further below.

It should be understood that the function $1/(1-x)$ is actually defined everywhere except at $x = 1$; for instance, it is defined at $x = 3$. But the function $1 + x + x^2 + x^3 + \cdots$ is defined only for those x where $|x| < 1$. So those two functions are *not* the same — they are defined on different domains.

All sorts of radii of convergence are possible. For instance, the series

$$\frac{1}{2} + \frac{x}{3 \cdot 1 + 2} + \frac{x^2}{9 \cdot 4 + 2} + \frac{x^3}{27 \cdot 9 + 2} + \frac{x^4}{81 \cdot 16 + 2} + \cdots + \frac{x^n}{3^n n^2 + 2} + \cdots$$

converges for $|x| \leq 3$ and diverges for $|x| > 3$, so this series has radius of convergence equal to 3.

To understand the general case, you first need to understand subsequences and limsups.

If you're given a sequence, and you cross out some of its terms — perhaps finitely many, perhaps infinitely many, but make sure that you still have infinitely many terms left — then what remains is called a **subsequence** of the given sequence. It might have a limit, even if the original sequence did not. For instance, $1, 0, 1, 0, 1, 0, \ldots$ does not have a limit, but

this crossing out	yields this subsequence	which converges to
$1, \not 0, 1, \not 0, 1, \not 0, 1, \not 0, \ldots$	$1, 1, 1, 1, 1, 1, 1, \ldots$	1
$1, \not 0, \not 1, \not 0, 1, \not 0, \not 1, \not 0, \ldots$	$1, 1, 1, 1, 1, 1, 1, \ldots$	1
$\not 1, 0, \not 1, 0, \not 1, 0, \not 1, 0, \ldots$	$0, 0, 0, 0, 0, 0, \ldots$	0
$1, 0, \not 1, 0, 1, 0, \not 1, 0, \ldots$	$1, 0, 0, 1, 0, 0, \ldots$	(no limit)
$\not 1, 0, 1, 0, 1, 0, 1, 0, \ldots$	$0, 1, 0, 1, 0, 1, \ldots$	(no limit)
$1, 0, \not 1, \not 0, \not 1, \not 0, \not 1, \not 0, \ldots$	$1, 0$ (not a subsequence)	

Now, it turns out that, if we allow $+\infty$ as a possible limit, then *every* sequence $a_1, a_2, a_3, a_4, \ldots$ of real numbers has at least one subsequence that converges. Furthermore, if you look at all the limits obtained in this fashion, there is a highest limit. That highest limit is called the **limsup** of the original sequence; it is denoted $\limsup_{n\to\infty} a_n$. Here are some examples:

- $1, 0, 1, 0, 1, 0, 1, 0, \ldots$ has limsup equal to 1.
- 1, 2, 1, 3, 1, 2, 1, 3, 1, 2, 1, 3, ... has limsup equal to 3.
- $0,\ -1,\ 1,\ 0,\ -2,\ 2,\ 0,\ -4,\ 3,\ 0,\ -8,\ 4,\ 0,\ -16,\ 5,\ 0,\ -32,\ 6,\ \ldots$ has limsup equal to $+\infty$.
- $1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \ldots$ has limsup equal to 0.

That last example illustrates an important point: if the original sequence *is* convergent to some limit, then all its subsequences are convergent to that same limit, and so that limit is also equal to the limsup.

Now we can talk about power series in general.

> **Theorem.** Suppose that $c_0, c_1, c_2, c_3, c_4, \ldots$ are some given complex numbers. Define the number $R = 1/\limsup_{n\to\infty} \sqrt[n]{|c_n|}$. Then R is a number in $[0,\infty]$, called the **radius of convergence** of the series $\sum_{n=0}^{\infty} c_n x^n$. It has the property that the series $\sum_{n=0}^{\infty} c_n x^n$ converges whenever we plug in a number x satisfying $|x| < R$, and diverges whenever $|x| > R$.

That's for *complex* numbers x, not just real numbers. So the numbers $\{x : |x| < R\}$ form the inside of a circle, and R is its radius; that's why R is called the radius of convergence. Actually, mathematicians use the term **disk of convergence** for the inside of the circle, and **circle of convergence** for the boundary of the circle.

There are some aspects of this theorem that still need explaining. Perhaps the most obvious question is, when happens when $|x| = R$? i.e., what happens along the circle? The answer is, all sorts of things could happen when $|x| = R$, and to explain them all we'd need a much more complicated theorem, so it's better (at least for beginners) if we simply don't try to draw any conclusion about that case. For instance, though I won't give proofs of these facts, it turns out that

$$\sum_{n=0}^{\infty} x^n \quad \text{converges for } |x| < 1, \quad \text{diverges for } |x| > 1, \quad \text{diverges for } |x| = 1;$$

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n n^2 + 2} \quad \text{converges for } |x| < 3, \quad \text{diverges for } |x| > 3, \quad \text{converges for } |x| = 3;$$

$$\sum_{n=0}^{\infty} \frac{2^n x^n}{n+1} \quad \text{converges for } |x| < \tfrac{1}{2}, \quad \text{diverges for } |x| > \tfrac{1}{2}, \quad \begin{cases} \text{converges for some } x\text{'s with} \\ |x| = \frac{1}{2} \text{ and diverges for others.} \end{cases}$$

Also, what does $R = 0$ or $R = \infty$ mean?

Well, take for instance $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$. This has $c_n = 1/n!$, It can be shown[1] that the numbers $\sqrt[n]{1/n!}$ converge to 0. So we get $R = \infty$. This means the series converges for every

[1] *Theorem.* $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$. *Outline of proof.* Let k be the highest integer less than or equal to $n/2$. That is, k is $n/2$ or $(n-1)/2$, according as n is even or odd. Then

$$n! \quad = \quad \underbrace{1 \cdot 2 \cdot \cdots \cdot (k-1) \cdot k}_{\substack{\text{these factors} \\ \text{are all } \geq 1}} \cdot \underbrace{(k+1) \cdot (k+2) \cdot \cdots \cdot (n-1) \cdot n}_{\substack{\text{there are } n/2 \text{ or more of these} \\ \text{factors; each is } n/2 \text{ or higher.}}} \quad \geq \quad 1^k \cdot \left(\frac{n}{2}\right)^{n/2}; \qquad \text{hence } \sqrt[n]{n!} \geq \sqrt{n/2}.$$

x satisfying $|x| < \infty$. That's true for every complex number x, so in fact the exponential series converges for every complex number.

On the other hand, consider the series $f(x) = 1 + x + 2!x^2 + 3!x^3 + 4!x^4 + \cdots$. This has $c_n = n!$. We have $\sqrt[n]{n!} \to \infty$, hence $R = 0$, which tells us that this series diverges for every complex number x that satisfies $|x| > 0$. The theorem doesn't tell us what happens when $|x| = 0$, but we can figure that out easily enough: When $x = 0$, then all of the numbers 1, $1 + x$, $1 + x + 2!x^2$, and so on are equal to 1, and so that sequence of partial sums converges to 1. Thus the series *does* converge when $x = 0$. We get

$$1 + x + 2!x^2 + 3!x^3 + 4!x^4 + \cdots \quad = \quad \begin{cases} 1 & \text{when } x = 0 \\ \text{undefined} & \text{otherwise.} \end{cases}$$

That's not a very useful function, evidently.

How do we *find* the radius of convergence?

Here is a rule that is usually a little simpler than the limsup formula.

> **Root test.** If the number $1/\lim_{n\to\infty} |\sqrt[n]{|c_n|}$ exists, then it is equal to $1/\limsup_{n\to\infty} \sqrt[n]{|c_n|}$, the radius of convergence.
>
> On the other hand, if $1/\lim_{n\to\infty} |\sqrt[n]{|c_n|}$ does *not* exist, then $R = 1/\limsup_{n\to\infty} \sqrt[n]{|c_n|}$ still *does* exist.

And here is a formula that is much simpler still:

> **Ratio test.** If the number $\lim_{n\to\infty} |c_n/c_{n+1}|$ exists, then it is equal to $1/\limsup_{n\to\infty} \sqrt[n]{|c_n|}$, the radius of convergence (and it may be easier to find).
>
> On the other hand, if $\lim_{n\to\infty} |c_n/c_{n+1}|$ does *not* exist, then $R = 1/\limsup_{n\to\infty} \sqrt[n]{|c_n|}$ still *does* exist.

Here is an example using the ratio test. One of the examples I gave earlier was $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{3^n n^2 + 2}$. In that example, $c_n = \dfrac{1}{3^n n^2 + 2}$, so we compute

$$\frac{c_n}{c_{n+1}} = \frac{3^{n+1}(n+1)^2 + 2}{3^n n^2 + 2} = \frac{3^n n^2 \left[3\left(1 + \frac{1}{n}\right)^2 + \dfrac{2}{3^n n^2}\right]}{3^n n^2 \left[1 + \dfrac{2}{3^n n^2}\right]} = \frac{3\left(1 + \frac{1}{n}\right)^2 + \dfrac{2}{3^n n^2}}{1 + \dfrac{2}{3^n n^2}} \to \frac{3 \cdot 1 + 0}{1 + 0}$$

so $R = 3$. Answering that problem with limsups would be difficult.

Usually the ratio test is easiest to use. And it's certainly easiest to explain. Consequently most textbooks don't even mention the other formulas. But that has a drawback: When they encounter a problem where the ratio test doesn't work, some students think that the radius of convergence "does not exist." That's wrong, and that's why I talk about limsups.

Here is an example where the ratio test doesn't work. I mentioned earlier that

$$\arctan x \quad = \quad x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots$$

Where is this equation valid? i.e., what is the radius of convergence of this series? In this example we have these coefficients and ratios of coefficients:

n	0	1	2	3	4	5	6	7	8	9	$\cdots$
c_n	0	1	0	$-1/3$	0	$1/5$	0	$-1/7$	0	$1/9$	$\cdots$
$\frac{c_n}{c_{n+1}}$	$\frac{0}{1}$	$\frac{1}{0}$	$\frac{0}{-1/3}$	$\frac{-1/3}{0}$	$\frac{0}{1/5}$	$\frac{1/5}{0}$	$\frac{0}{-1/7}$	$\frac{-1/7}{0}$	$\frac{0}{1/9}$	$\frac{1/9}{0}$	$\cdots$

In that last row, we're alternating between 0 and undefined (since any fraction with 0 in its bottom is meaningless). If we omit all the undefined terms, what is left is all zeros, and the limit of 0 is 0, which is the wrong answer for the radius of convergence. If we include the undefined terms, then we don't know how to take the limits at all. The ratio limit doesn't exist, but that does *not* mean the radius of convergence doesn't exist.

Let's try using the root test instead.

n	0	1	2	3	4	5	6	7	8	9	$\cdots$
$\|c_n\|$	0	1	0	1/3	0	1/5	0	1/7	0	1/9	$\cdots$
$\sqrt[n]{\|c_n\|}$	?	1	0	$\sqrt[3]{1/3}$	0	$\sqrt[5]{1/5}$	0	$\sqrt[7]{1/7}$	0	$\sqrt[9]{1/9}$	$\cdots$

where the question mark means "I don't know"; the zeroth root of anything is not defined, but the first term (or first finitely many terms) has no effect on the limit of a sequence. Now, look in your calculus book, in the chapter on l'Hospital's Rule and indeterminate limits; you'll find computations like this: $\lim_{n\to\infty} \sqrt[n]{1/n} = \lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = \lim_{x\to 0} x^x = \lim_{x\to 0} e^{x \ln x} = e^0 = 1$. So that tells us that the sequence

$$1,\ \sqrt[2]{1/2},\ \sqrt[3]{1/3},\ \sqrt[4]{1/4},\ \sqrt[5]{1/5},\ \sqrt[6]{1/6},\ \sqrt[7]{1/7},\ \ldots$$

converges to 1. On the other hand, the sequence $0, 0, 0, \ldots$ converges to 0. Alternating those two sequences gives us the sequence $\sqrt[n]{|c_n|}$. Therefore that sequence doesn't converge, but it has subsequences converging to 0 and 1 (and if you think about it a bit, you'll see that no other limits are possible). Of those, the higher number is 1. Therefore $\limsup_{n\to\infty} \sqrt[n]{|c_n|} = 1$, and $R = 1/1 = 1$. The radius of convergence is 1.

Actually, this problem could also be done using the ratio test, with less computation and more understanding, if you have a good mind for abstract theory. But the following method is rather specialized — it only works in a few problems. Substitute $u = x^2$, and we can write $\arctan x = x\left(1 - \frac{u}{3} + \frac{u^2}{5} - \frac{u^3}{7} + \frac{u^4}{9} - \cdots\right)$. Now, for the moment, forget about the factor of x in front. For which values of u does the series $1 - \frac{u}{3} + \frac{u^2}{5} - \frac{u^3}{7} + \frac{u^4}{9} - \cdots$ converge? This series has

n	0	1	2	3	4	5	$\cdots$
$\|c_n\|$	1	1/3	1/5	1/7	1/9	1/11	$\cdots$
$\|c_n/c_{n+1}\|$	3	5/3	7/5	9/7	11/9	13/11	$\cdots$

The numbers in that last row converge to 1. Therefore the series $1 - \frac{u}{3} + \frac{u^2}{5} - \frac{u^3}{7} + \frac{u^4}{9} - \cdots$ converges whenever $|u| < 1$, and diverges whenever $|u| > 1$. Therefore the series for arctan converges whenever $|x^2| < 1$ and diverges whenever $|x^2| > 1$. Therefore the series for arctan converges whenever $|x| < 1$ and diverges whenever $|x| > 1$. So its radius of convergence is 1.