INTRODUCTION TO COMPLEX NUMBERS

Introduction to the introduction: Why study complex numbers? Well, complex numbers are the best way to solve polynomial equations, and that's what we sometimes need for solving certain kinds of differential equations. For instance,

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} = 0$$

is a constant-coefficient homogeneous linear ordinary differential equation; we want to find $y(t)$. (Don't worry about the terminology right now; the terminology isn't the main point of this discussion.) There is a theorem that tells us that any equation of that type has at least one solution of the form $y = e^{kt}$ for some constant k. (Actually, the theory tells us quite a bit more, but the other things it tells us are more complicated and so they don't belong in this introduction.) Now, we just need to find k. To do that, compute:

$$y = e^{kt}, \qquad y' = ke^{kt}, \qquad y'' = k^2e^{kt}, \qquad y''' = k^3e^{kt}.$$

Substitute those into the differential equation; that yields the equation

$$k^3e^{kt} + 6k^2e^{kt} + 5ke^{kt} = 0.$$

Since e^{kt} is never 0, we can divide it out. That leaves the polynomial equation

$$k^3 + 6k^2 + 5k = 0.$$

That has solutions $k = 0$, $k = -1$, and $k = -5$. All the preceding steps are reversible, so three solutions of the differential equation are 1, e^{-t}, e^{-5t}. In fact, using other parts of the general theory (not discussed here), it can be shown that the *general* solution of this particular differential equation is

$$y = c_0 + c_1e^{-t} + c_2e^{-5t}$$

where c_0, c_1, c_2 are arbitrary constants.

This method of computation works for *any* constant-coefficient homogeneous linear ordinary differential equation, and a variant of this method works for nonhomogeneous equations too. But there are certain complications we need to deal with:

1. What if the resulting polynomial is one with no real solutions? For instance, $y''+y = 0$ yields the polynomial equation $k^2 + 1 = 0$, which has no real solutions. And the solution of $y'' + y = 0$ turns out to be $c_0 \sin t + c_1 \cos t$. What is the general method for equations like this?

2. What if the polynomial has repeated roots? For instance, $y'' + 2y + 1 = 0$ yields the polynomial equation $k^2 + 2k + 1 = 0$, which factors to $(k + 1)^2 = 0$. In this case it turns out that the differential equation has general solution given not by $c_0e^{-t} + c_1e^{-t}$, but rather by $c_0e^{-t} + c_1te^{-t}$. What is the general method?

The theory of complex numbers is very deep. Graduate students in mathematics may take a semester or two of courses just devoted to the deeper properties of complex numbers. However, all we'll need for our differential equations is the material covered in the first few days of those courses.

What are complex numbers? In calculus we used the **real number system**, $\mathbb{R}$; we describe it as the points on a line. However, for some purposes — e.g., polynomials and differential equations — we need more numbers. The **complex number system**, $\mathbb{C}$, can be described as the points on a *plane*.

Of course, you've worked with points on a plane before. But we'll be using a notation that may be different from what you've used before. More importantly, we'll define *multiplication* of two points in the plane in a fashion that may be different from what you've had before.

The point $(3, 5)$ is 3 units to the right of the origin and 5 units up. But it is customary to change notation at this point: When we think of that point as a complex number, we write it instead as $3 + 5i$. This emphasizes that it is viewed as just *one* "number," rather than a pair of real numbers. (Some engineers write j instead of i.)

Some abbreviations are possible: the number $0 + 5i$ (which lies on the vertical coordinate axis) may be written more briefly as $5i$, and the number $3 + 0i$ (which lies on the horizontal axis) may be written simply as 3. That is, the complex numbers on the horizontal axis are just ordinary real numbers; the real numbers are a subset of the complex numbers. That is, $\mathbb{R} \subseteq \mathbb{C}$, where $\mathbb{R} = \{\text{real numbers}\}$ and $\mathbb{C} = \{\text{complex numbers}\}$.

There is also some peculiar terminology: the numbers on the horizontal axis are said to be **real**, and the numbers on the vertical axis (such as $5i$) are said to be **imaginary**. In a number such as $3 + 5i$, we say 3 is the **real part** and 5 is the **imaginary part**. The reason for this terminology is explained on a later page.

You've worked with points in the plane before. What may be new to you are the *arithmetic rules* for the complex numbers.

How to add two complex numbers: It's done coordinatewise, like the addition of two-dimensional vectors:

$$(a + bi) \ + \ (c + di) \quad = \quad (a + c) \ + \ (b + d)i.$$

For instance, $(3 + 5i) + (7 - 2i) = 10 + 3i$, and $(6 + \pi i) + (\sqrt{2} - 7i) = (6 + \sqrt{2}) + (\pi - 7)i$. That might look more familiar if we use the ordered pair notation:

$$(a, b) + (c, d) = (a + c, b + d); \qquad (3, 5) + (7, -2) = (10, 3), \qquad \text{etc.}$$

But we will use the "i" notation instead. — Subtraction is similar; we subtract coordinatewise. — Of course, this is all with *Cartesian* coordinates. We could do our addition or subtraction with polar coordinates, but it would then be much more complicated.

How to multiply two complex numbers: This is more complicated, and it involves a formula you'll need to memorize:

$$\boxed{(a + bi)(c + di) \quad = \quad (ac - bd) + (ad + bc)i}\,.$$

I may refer to this as the **Cartesian formula for multiplication**, to contrast it with the *polar* formula which will be given later.

The Cartesian formula admittedly looks rather contrived and arbitrary; one might ask *why* anyone would want to define a "multiplication" operation in this fashion. Trust me on that issue for a few paragraphs; the reasons for this formula will be given later.

Here is an example: $(3+5i)(7-2i) = (3 \cdot 7 - 5 \cdot (-2)) + (3 \cdot (-2) + 5 \cdot 7)i = 31 + 29i$.

A more surprising example is $(0+i)(0+i) = (-1+0i)$. More briefly,

$$i^2 = -1.$$

And we also have $(-i)^2 = -1$. Thus, the number -1 has two square roots.

Now you can see where the term "imaginary" came from: It is a historical accident. Complex numbers were first investigated algebraically; the points-in-the-plane geometric model only came many decades later. The first mathematicians who worked with the number i were so surprised by its properties that they thought "there can't *really* be a number with these properties." They worked out the computational properties of a collection of numbers in whose existence they didn't believe. They found these "numbers" to be a convenient fiction, a useful intermediate step in going from a problem involving only real numbers to an answer involving only real numbers. They called the number i "imaginary," and unfortunately that name stuck.

Here's a bit of philosophy: In some sense, *all* numbers are objects that exist only in the imagination. The physical world that we live in may contain three airplanes or three apples, but it does not contain "three"; that is an abstract notion that only exists in our minds. The abstract notion is useful because it enables us to understand the physical world better. Positive numbers were good for trading sheep and goats; negative numbers expanded our language so that we could also discuss debts and cold temperatures. Finally, complex numbers are particularly good for explaining rotations and stretching motions, as we shall see below.

Absolute values. The absolute value of a complex number $\alpha = p + qi$ is the number $|\alpha| = |p + qi| = \sqrt{p^2 + q^2}$. It is the Euclidean distance from 0 to $p + qi$, which can be computed using the Pythagorean Theorem.

Exercises. Simplify — i.e., write the product in the form $p + qi$, where p and q are real numbers. Also, find the absolute value of the product.

(1) $(3+4i)(5+6i)$.

(2) $(5-3i)(2+0i)$.

(3) $(2-\sqrt{5}i)(2+\sqrt{5}i)$.

(*Optional.*) If you're already familiar with matrix multiplication, then you might like this explanation for complex multiplication: Represent the complex number $x + yi$ with the 2-by-2 matrix of real numbers $\begin{bmatrix} x & y \\ -y & x \end{bmatrix}$. Then it turns out that addition of complex numbers is the same as addition of 2-by-2 matrices, and multiplication of complex numbers is the same as multiplication of 2-by-2 matrices. For instance, our example $(3+5i)(7-2i) = 31+29i$ can be restated as

$$\begin{bmatrix} 3 & 5 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 7 & -2 \\ 2 & 7 \end{bmatrix} \quad = \quad \begin{bmatrix} 31 & 29 \\ -29 & 31 \end{bmatrix}.$$

However, the real motivation for complex multiplication — i.e., the *real* reason why we want to define complex multiplication this way — will be apparent when we discuss polar coordinates, a few paragraphs from now.

Basic rules of arithmetic. Complex numbers obey many of the same familiar rules that you already learned for real numbers. For instance, for any complex numbers α, β, γ, we have

- Commutative laws: $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$.
- Associative laws: $(\alpha + \beta) + \gamma = \gamma + (\beta + \gamma)$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.
- Distributive law: $\alpha(\beta + \gamma) = (\alpha\beta) + (\alpha\gamma)$.
- Identity laws: $\alpha + 0 = \alpha$ and $\alpha \cdot 1 = \alpha$.

When a and b are real numbers, the additive inverse of $a + bi$ is $(-a) + (-b)i$, since $(a + bi) + ((-a) + (-b)i) = 0$.

If a and b are not both zero, then $a + bi$ also has a multiplicative inverse, but it's a bit complicated — it is

$$\frac{a}{a^2+b^2} \quad + \quad \frac{-b}{a^2+b^2}\, i.$$

Try multiplying that times $a + bi$; you should get 1 as the result. Note that the condition "a and b are not both zero" is the same as the condition "the complex number $a+bi$ is not the complex number 0." Thus, every complex number other than 0 has a multiplicative inverse.

For instance, $(6 + 8i) \cdot (0.06 - 0.08i) = 1$. So we can write $\dfrac{1}{6+8i} = 0.06 - 0.08i$. And so $\dfrac{3-i}{6+8i} = (3-i)(.06 - .08i) = 0.1 - 0.3i$.

The preceding rules — commutative, associative, existence of inverses, etc. — are the rules for a *field*, in abstract algebra. Thus the complex numbers are a field, just as the rational numbers and the real numbers are fields.

However, the rationals and reals are *ordered* fields, and the complex numbers are not. In fact, no matter how cleverly we set up the definition, it is *not possible* to define an ordering relation $<$ on $\mathbb{C}$ that has all the usual properties of the ordering of the reals. (Indeed, from the ordering properties of the reals it can be proved algebraically that $x^2 + 1 = 0$ has no solution. Since $\mathbb{C}$ does have a solution for that equation, $\mathbb{C}$ can't satisfy all those ordering properties.)

Exercises. Simplify; write each number in the form $a + bi$ where a and b are real numbers that are simplified as much as possible.

(4) $\dfrac{1}{2+i}$

(5) $\dfrac{7-3i}{2+i}$

Solving polynomial equations with complex numbers. When we have more numbers, we can solve more problems. Some problems have solutions that can only be expressed in terms of the new numbers. For instance, we've already mentioned that $i^2 = -1$, so $z = i$ is a solution of the equation $z^2 = -1$, or $z^2 + 1 = 0$. This problem didn't have a solution in the real number system, but it does in the complex number system. Indeed, complex numbers enable us to solve *any* polynomial equation (at least, in principle). We have:

> **Fundamental Theorem of Algebra.** Let any polynomial be given:
>
> $$p(z) \;=\; a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0$$
>
> where n is a positive integer, and $a_0, a_1, a_2, \ldots, a_n$ are complex numbers, with $a_n \neq 0$. Then that polynomial can also be written
>
> $$p(z) \;=\; a_n (z - r_1)(z - r_2) \cdots (z - r_{n-1})(z - r_n)$$
>
> for some complex roots $r_1, r_2, r_3, \ldots, r_n$ (not necessarily all different).

The theorem above doesn't tell us how to find the roots; it just tells us the roots *exist*. In some cases it is hard to find the roots. There are methods for using computers to approximate the roots (with as much accuracy as we wish), but that is a complicated subject which will not be addressed here. Instead we'll just look at methods for getting exact solutions to some of the easier polynomial equations.

For polynomials of degree two, finding the roots is easy: Just use the **quadratic formula**. The roots of $az^2 + bz + c = 0$ are

$$z \quad = \quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This is the same formula that you learned in high school (and it has the same proof), but it has a new meaning now: We no longer require $b^2 - 4ac$ to be a positive number or zero.

- If $b^2 - 4ac$ is positive, then $\sqrt{b^2 - 4ac}$ is also positive, and we obtain two different solutions.
- If $b^2 - 4ac = 0$, then $\sqrt{b^2 - 4ac} = 0$, and we obtain a repeated root.
- If $b^2 - 4ac < 0$, then $\sqrt{4ac - b^2}$ is a positive number, and

  $$\pm\sqrt{b^2 - 4ac} = \pm\sqrt{(-1)(4ac - b^2)} = \pm i\sqrt{4ac - b^2},$$

  so we obtain two solutions: $z = [-b \pm \sqrt{b^2 - 4ac}]/(2a)$.

- And there is still another possibility: If a, b, c are complex numbers and they are not all real numbers, then $b^2 - 4ac$ might not be a real number either. In this case it turns out that $b^2 - 4ac$ has two different square roots (though finding them is a little complicated; we'll discuss that a few paragraphs from now). So we still get two solutions: $z = [-b \pm \sqrt{b^2 - 4ac}]/(2a)$.

We can also solve quadratic equations by completing the square:

$$az^2 + bz + c \;=\; 0 \quad\Rightarrow\quad z^2 + \frac{b}{a}z + \frac{c}{a} = 0 \quad\Rightarrow\quad z^2 + \frac{b}{a}z + \frac{b^2}{4a^2} \;=\; \frac{b^2}{4a^2} - \frac{c}{a} \quad\Rightarrow$$

$$\left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad\Rightarrow\quad z + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \quad\Rightarrow\quad z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In some problems, this procedure is easier than the quadratic formula, because it yields simplifications along the way.

Example. Solve $z^2 + 4z + 5 = 0$. *Solution.* Use the quadratic formula with $a = 1$, $b = 4$, $c = 5$. We get

$$z = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i.$$

This can also be solved by completing the square:

$$z^2 + 4z + 4 + 1 = 0 \quad\Rightarrow\quad (z+2)^2 - (-1) = 0 \quad\Rightarrow\quad (z+2)^2 - (i)^2 = 0$$

$$\Rightarrow\quad (z + 2 + i)(z + 2 - i) = 0 \quad\Rightarrow\quad z = -2 \pm i.$$

Example: Solve $z^3 + 5z^2 + 6z - 12 = 0$. *Solution*: Since this is a "textbook problem" and you're not expected to know how to solve all third degree equations, it must have at least one easy solution. Try a few small integers; you'll find right away that $z = 1$ is a solution. (You could also find that on your calculator.) Now divide out $z - 1$. You'll find $z^3 + 5z^2 + 6z - 12 = (z - 1)(z^2 + 6z + 12)$. Thus, it remains for us to solve the quadratic equation $z^2 + 6z + 12$. By the quadratic formula or by completing the square, we find that that problem has solutions $-3 \pm \sqrt{3}i$. Thus, the three solutions are 1, $-3 + \sqrt{3}i$, and $-3 - \sqrt{3}i$.

Exercises. Solve each equation. Write the answers in the form $a + bi$ where a and b are real numbers that are simplified as much as possible. In most cases, a quadratic equation should have two answers.

(6) $z^2 + 10z + 100 = 0$.

(7) $z^2 + 6iz - 9 = 0$.

(8) $z^2 + 10z + 25 = 0$.

(9) $z^2 + 10z + 16 = 0$.

(10) $z^2 - 8z + 36 = 0$.

(11) $z^3 + 10z^2 + 100z = 0$.

(12) $z^3 + 5z^2 + 15z + 11 = 0$.

Polar coordinates. For any real numbers a and b, we can find some real numbers θ and r (generally with $r \geq 0$) such that $(a, b) = (r\cos\theta, r\sin\theta)$. In other words, $a + bi = r\cos\theta + ir\sin\theta$. We have $r = |a + bi| = \sqrt{a^2 + b^2}$, and θ is one of the numbers that satisfies

$$\cos\theta = \frac{a}{\sqrt{a^2 + b^2}} \qquad \text{and} \qquad \sin\theta = \frac{b}{\sqrt{a^2 + b^2}}, \qquad \text{hence} \qquad \tan\theta = \frac{b}{a}.$$

That number θ is not unique — we can add any multiple of 2π to any acceptable value for θ, and get another acceptable value for θ.

Caution. Some people may be tempted to say that θ can be defined by one of the equations

$$\cos\theta = \frac{a}{\sqrt{a^2 + b^2}} \qquad \text{or} \qquad \sin\theta = \frac{b}{\sqrt{a^2 + b^2}} \qquad \text{or} \qquad \tan\theta = \frac{b}{a},$$

but that's not quite right. Each of those equations is satisfied by θ, but no one of those equations is enough to determine θ.

For instance, for the complex number $1 + \sqrt{3}i$, the correct angle is $\theta = \pi/3$. It is true that we have

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \qquad \sin\left(\frac{\pi}{3}\right) = \frac{1}{2}\sqrt{3}, \qquad \tan\left(\frac{\pi}{3}\right) = \sqrt{3}.$$

But no *one* of those equations by itself gives enough information. Indeed, we also have

$$\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}, \qquad \sin\left(\frac{2\pi}{3}\right) = \frac{1}{2}\sqrt{3}, \qquad \tan\left(\frac{4\pi}{3}\right) = \sqrt{3},$$

but the numbers $-\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$ are three different *wrong* answers for the angle of $1 + \sqrt{3}i$. Those three angles yield quadrants IV, II, III respectively, while the point $1 + \sqrt{3}i$ lies in quadrant I,

Multiplication revisited. Now, why would we want to use

$$(a + bi)(c + di) \;=\; (ac - bd) + (ad + bc)i$$

as our definition of the multiplication of complex numbers? That rule is not extraordinarily difficult, but it does seem rather arbitrary. The real advantages of that definition become apparent when we switch to polar coordinates. Suppose that

$$a + bi \;=\; r(\cos\theta + i\sin\theta) \qquad \text{and} \qquad c + di \;=\; s(\cos\varphi + i\sin\varphi).$$

Substitute those into the Cartesian definition of multiplication, and then simplify the results using the angle-addition identities from trigonometry. We end up with

$$r(\cos\theta + i\sin\theta)\, s(\cos\varphi + i\sin\varphi) \quad = \quad (rs)[\cos(\theta + \varphi) + i\sin(\theta + \varphi)].$$

This is the **polar formula for multiplication**. It says that when we multiply complex numbers, we add the angles and multiply the radii.

When we multiply a collection of complex numbers by some particular complex number $r\cos\theta + ir\sin\theta$, the effect is to rotate the plane through an angle of θ and stretch (or shrink) the plane by a magnification factor of r. This effect is of great importance in physics and engineering: there are many physical processes that involve rotation and/or stretching — e.g., the rotation of a motor, the rotation of a star. Such processes can be expressed and computed more easily if we use complex numbers in our representation. There is nothing "imaginary" about this; these processes are very real and concrete.

Example: Convert these numbers to polar coordinates, multiply, and then convert the answer back to Cartesian coordinates: $3i \cdot (\frac{5}{2} + \frac{5}{2}\sqrt{3}i)$.

Solution. $3i \cdot (\frac{5}{2} + \frac{5}{2}\sqrt{3}i) = (3\cos\frac{\pi}{2} + 3i\sin\frac{\pi}{2})\,(5\cos\frac{\pi}{3} + 5i\sin\frac{\pi}{3}) = 15\cos\frac{5\pi}{6} + 15i\sin\frac{5\pi}{6} = \frac{-15}{2}\sqrt{3} + \frac{15}{2}\sqrt{3}i.$

Exercise. Multiply this out, and then convert your answer to Cartesian coordinates — i.e., write in the form $p + qi$, where p and q are real numbers, simplified as much as possible. Your answer should not involve sine or cosine.

$$\left(3\cos\frac{\pi}{4} + 3i\sin\frac{\pi}{4}\right)\left(5\cos\frac{2\pi}{3} + 5i\sin\frac{\pi}{3}\right). \tag{13}$$

Powers of complex numbers. What happens in polar coordinates, when we multiply a complex number times *itself* one or more times? Again, we multiply the radii and add the angles. We get

$$\begin{aligned}(r\cos\theta + ir\sin\theta)^2 &= r^2\cos(2\theta) + ir^2\sin(2\theta),\\ (r\cos\theta + ir\sin\theta)^3 &= r^3\cos(3\theta) + ir^3\sin(3\theta),\\ (r\cos\theta + ir\sin\theta)^4 &= r^4\cos(4\theta) + ir^4\sin(4\theta),\end{aligned}$$

and in general

$$(r\cos\theta + ir\sin\theta)^n = r^n\cos(n\theta) + ir^n\sin(n\theta).$$

This is **De Moivre's formula**. *To raise a complex number to the power n, where the complex number is already given in polar coordinates, just raise its radius to the power n, and multiply its angle by n.*

Here is an example: Find the number $(2+2i)^7$. The most obvious method is to use the Cartesian formula for multiplication, but multiplying everything out is very tedious. An easier method is to first rewrite $2+2i$ in polar form. The point $(2,2)$ has distance from the origin equal to $2\sqrt{2}$ (by the Pythagorean Theorem), and the line segment from $(0,0)$ to $(2,2)$ has slope 1, which has arctangent equal to $\pi/4$. Thus

$$2 + 2i = 2\sqrt{2}\cos\frac{\pi}{4} + 2\sqrt{2}i\sin\frac{\pi}{4} = 2^{3/2}\cos\frac{\pi}{4} + 2^{3/2}i\sin\frac{\pi}{4}.$$

Then $(2+2i)^7 = (2^{3/2})^7 \cos\frac{7\pi}{4} + (2^{3/2})^7 i \sin\frac{7\pi}{4}$. Note that $\cos\frac{7\pi}{4} = \frac{1}{2}\sqrt{2} = \frac{1}{\sqrt{2}}$ and $\sin\frac{7\pi}{4} = \frac{-1}{2}\sqrt{2} = \frac{-1}{\sqrt{2}}$. Also, we have $(2^{3/2})^7 = 2^{21/2} = 2^{10}\sqrt{2} = 1024\sqrt{2}$. Then $(2+2i)^7 = 1024 - 1024i$; that's the answer.

Exercise. Simplify — i.e., write in the form $p + qi$, where p and q are real numbers. Your answer should not involve sine or cosine.

(14) $(\sqrt{3}+i)^6$

(15) $(-i)^9$

Roots of complex numbers. To find square roots, cube roots, etc., we can reverse that last process — i.e., we can reverse De Moivre's formula. *To find the nth roots of a complex number that is already given in polar coordinates, just raise its radius to the power $1/n$, and divide its angle by n.* However, there is a new complication, that was not present in the nth power situation: Each complex number has just one nth root (if n is a positive integer), but **each complex number has n different nth roots**[1]. How do we find those n nth roots?

Say we're looking for nth roots of $s\cos\psi + is\sin\psi$. First note that

$$\begin{aligned} s\cos\psi + is\sin\psi \quad &= \quad s\cos(\psi+2\pi) + is\sin(\psi+2\pi) \quad = \\ &\cdots \quad = \quad s\cos(\psi+2(n-1)\pi) + is\sin(\psi+2(n-1)\pi). \end{aligned}$$

(We could continue that sequence further, but n angles are enough, as explained below.)

Now, those are n different representations of the *same* point in the complex plane, since those angles differ by multiples of 2π. But when we divide the angles by n and replace the radii with their nth roots, we obtain n *different* points in the complex plane, since the resulting angles differ by multiples of $2\pi/n$:

$$\sqrt[n]{s}\cos\psi + i\sqrt[n]{s}\sin\psi\,, \qquad \sqrt[n]{s}\cos\left(\psi+\frac{2\pi}{n}\right) + i\sqrt[n]{s}\sin\left(\psi+\frac{2\pi}{n}\right),$$
$$\cdots, \qquad \sqrt[n]{s}\cos\left(\psi+\frac{2(n-1)\pi}{n}\right) + i\sqrt[n]{s}\sin\left(\psi+\frac{2(n-1)\pi}{n}\right).$$

In summary,

> If $a+bi$ has angle ψ, then the n nth roots of $a+bi$ are
>
> $$\sqrt[n]{|a+bi|}\cos\left(\frac{\psi+2k\pi}{n}\right) + i\sqrt[n]{|a+bi|}\sin\left(\frac{\psi+2k\pi}{n}\right) \qquad (k = 0, 1, 2, \ldots, n-1)$$

Those n numbers are equally spaced along a circle, centered at the origin, with radius $\sqrt[n]{s}$. Actually, we don't have to let k run through the numbers $0, 1, 2, \ldots, n-1$; we could instead

[1]Exception: Zero has only one nth root.

let k run through $1, 2, 3, \ldots, n$, or, indeed, through any set of n consecutive integers. It should now be evident why we did not continue our list of angles $\psi, \psi + 2\pi, \psi + 4\pi, \ldots$ any further: After the first n answers, our answers would start repeating.

Here is an example: Let's find the four fourth roots of $-2 + 2\sqrt{3}i$. Here $a = -2$ and $b = 2\sqrt{3}$, so $|a + bi| = \sqrt{a^2 + b^2} = \sqrt{4 + 12} = 4$. Then $b/a = -\sqrt{3}$, which has arctangent equal to $-\pi/3$.

Note that the arctangent alone is *not* quite enough to tell us what angle we need for ψ; we cannot jump to the conclusion that $\psi = -\pi/3$. Indeed, *both* of the points $2 - 2\sqrt{3}i$ and $-2 + 2\sqrt{3}i$ yield arctangent equal to $-\pi/3$, but those two points are in different quadrants; their angles differ by π. The point we're interested in is $-2 + 2\sqrt{3}i$, which is in the upper left quadrant of the plane, so we need $\frac{\pi}{2} \leq \psi \leq \pi$. To get that result, take the number $-\pi/3$ and add π; that gives us $\psi = 2\pi/3$.

Thus, we have found that $-2 + 2\sqrt{3}i = 4\cos\left(\frac{2\pi}{3}\right) + 4i\sin\left(\frac{2\pi}{3}\right)$. The formula now tells us immediately that the four fourth roots of $-2 + 2\sqrt{3}i$ are

$$\sqrt[4]{4}\cos\left(\frac{\frac{2\pi}{3} + 2k\pi}{4}\right) + \sqrt[4]{4}i\sin\left(\frac{\frac{2\pi}{3} + 2k\pi}{4}\right) \qquad (k = 0, 1, 2, 3).$$

Let's simplify that a bit. Note that $\sqrt[4]{4} = \sqrt{2}$. Thus the four fourth roots are

$$\sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right), \qquad \sqrt{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right),$$

$$\sqrt{2}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right), \qquad \sqrt{2}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right).$$

Now make use of these values of the trigonometric functions:

α	$\cos\alpha$	$\sin\alpha$
$\pi/6$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$
$2\pi/3$	$\frac{-1}{2}$	$\frac{1}{2}\sqrt{3}$
$7\pi/6$	$\frac{-1}{2}\sqrt{3}$	$\frac{-1}{2}$
$5\pi/3$	$\frac{1}{2}$	$\frac{-1}{2}\sqrt{3}$

Simplifying, we find that the four fourth roots of $-2 + 2\sqrt{3}i$ are

$$\frac{1}{2}\sqrt{6} + \frac{1}{2}\sqrt{2}i, \qquad \frac{-1}{2}\sqrt{2} + \frac{1}{2}\sqrt{6}i, \qquad \frac{-1}{2}\sqrt{6} + \frac{-1}{2}\sqrt{2}i, \qquad \frac{1}{2}\sqrt{2} + \frac{-1}{2}\sqrt{6}i.$$

We can check these answers using the Cartesian formula for multiplication.

In some problems, an easier method is available — we can just solve a polynomial equation. For instance, to find the three cube roots of 8, start with the polynomial equation $z^3 - 8 = 0$. We already know that one of the three cube roots is 2, so factor it out; we get $(z - 2)(z^2 + 2z + 4) = 0$. Now solve $z^2 + 2z + 4 = 0$, either by the quadratic formula or by completing the square. We obtain $z = -1 \pm \sqrt{3}i$. Thus the three cube roots of 8 are $\boxed{2,\ -1+\sqrt{3}i, \text{ and } -1-\sqrt{3}i}$. As a partial check of your answer, you might sketch all three of those points on one graph (keeping in mind that $\sqrt{3}$ is approximately 1.7); those three points should be equally spaced along a circle centered at the origin.

We could get the same answer by the trigonometric method. The number 8 has radius 8 and angle 0. Its three cube roots should have radius 2 and angles 0, $2\pi/3$, and $4\pi/3$. Thus the three cube roots are

$$2\cos(0)+2i\sin(0), \qquad 2\cos\left(\frac{2\pi}{3}\right)+2i\sin\left(\frac{2\pi}{3}\right), \qquad 2\cos\left(\frac{4\pi}{3}\right)+2i\sin\left(\frac{4\pi}{3}\right).$$

Simplify those three numbers; you get the three boxed numbers of the previous paragraph.

Exercises. Evaluate and simplify:

(16) What are the two square roots of $2i$?

(17) What are the four fourth roots of 1 ?

(18) What are the three cube roots of -1 ?

Historical remarks: To a large extent, complex numbers were discovered in the 16th century, when mathematicians were trying to solve cubic and quartic equations. However, complex numbers were not really understood until the points-in-a-plane idea developed, a couple of hundred years later.

The quadratic formula was known, in a less abstract form, thousands of years ago. The cubic formula was found in the 16th century, but we generally don't teach it to our college students, because it is too complicated. Here it is: Divide out the coefficient of z^3; thus we can assume that the problem we're trying to solve is $z^3+bz^2+cz+d=0$. Then the solution is

$$\begin{aligned} z \quad = \quad & \sqrt[3]{-\frac{b^3}{27}+\frac{bc}{6}-\frac{d}{2}+\sqrt{\left(-\frac{b^3}{27}+\frac{bc}{6}-\frac{d}{2}\right)^2+\left(\frac{c}{3}-\frac{b^2}{9}\right)^3}} \\ & + \sqrt[3]{-\frac{b^3}{27}+\frac{bc}{6}-\frac{d}{2}-\sqrt{\left(-\frac{b^3}{27}+\frac{bc}{6}-\frac{d}{2}\right)^2+\left(\frac{c}{3}-\frac{b^2}{9}\right)^3}} - \frac{b}{3}. \end{aligned}$$

That looks like six solutions, since there are 3 cube roots and 2 square roots, but actually there is some repetition involved, and we only end up with 3 solutions. (I will *not* require you to memorize this formula. I just thought you might like to see it once, for curiosity's sake.)

For polynomials of degree 4, there is also an analogous formula, but it is much much more complicated, so I won't write it down here. For polynomials of degree 5 or higher, there *isn't* an analogous formula; that was proved by Abel in 1826. Still, the Fundamental Theorem of Algebra tells us that the solutions *exist.* We can approximate them using modern computers.

I don't just mean that no one has found the formula yet; I mean that Abel proved there *can't* be a formula of this sort. In other words, you can't express the solution of the general fifth degree polynomial equation using just the coefficients of the polynomial together with addition, subtraction, multiplication, division, and $\sqrt{\ }$, $\sqrt[2]{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, $\sqrt[5]{\ }$. Those simply aren't enough functions. If you have those functions as buttons on a calculator, then you don't have a fancy enough calculator; you need at least one more button to do 5th degree polynomials. There are a few different ways to choose the additional function; my favorite is this: Get a button labeled "f^{-1}" for the *inverse* of the function $f(x)=x^5+x$. It *is* possible to write down an exact formula for the general 5th degree polynomial, just involving the functions mentioned above (including f^{-1}); but that formula is horribly horribly complicated; we don't even want to think about it.

Extending the basic transcendental functions. There are several ways to define the cosine function. You probably learned it first in terms of a unit circle. If you draw an angle θ from the horizontal axis, then $\cos\theta$ is the distance in the x-direction, etc. But here is another way to obtain that same cosine function:

$$\cos z \quad = \quad 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \cdots.$$

This is an *infinite series.* Remember what that means: You don't really try to add up infinitely many terms; that would take too long. Just add up the first 100 terms or the first million terms or something like that, and you'll get a good approximation to what we mean by the "sum" of the series.

This whole discussion assumes you're using *radians* for your measurement of angles. If you use degrees, you'll have to replace the formulas given here with other, much more complicated formulas. Mathematicians prefer radians, and this preference is not arbitrary; radians make many of our formulas work out much more simply.

The unit circle explanation only defines $\cos(z)$ when z is a real number. But the infinite series makes sense if z is any *complex* number! The two definitions give the same result when z is real, but the infinite series is more convenient for differential equations.

And this approach also works for the functions $\sin z$ and e^z. Here are the formulas for those functions:

$$\sin z \quad = \quad z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \frac{z^{11}}{11!} + \cdots$$

$$e^z \quad = \quad 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots$$

This gives us a *natural* way of *extending* these functions — they still take their familiar old values when z is a real number, but the functions now are defined for all z in the complex plane. And they preserve many of their familiar properties — e.g.,

$$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$$

$$\sin(\theta + \varphi) = \cos\theta\sin\varphi + \sin\theta\cos\varphi$$

$$e^{\theta+\varphi} = e^\theta e^\varphi$$

for any *complex* numbers θ and φ. Also,

$$\frac{d}{dz}\sin z = \cos z, \qquad \frac{d}{dz}\cos z = -\sin z, \qquad \frac{d}{dz}e^z = e^z.$$

We also get some new formulas, which may come as a bit of a surprise:

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad e^{-i\theta} = \cos\theta - i\sin\theta, \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Because of those last few formulas, we can do all of our differential equations in terms of exponentials, without using sines and cosines; this turns out to be a big simplification in our reasoning.

Transforming complex solutions to real solutions. The theory of constant coefficient differential equations yields solutions involving terms such as e^{kt}, where the constant k is a *complex* number. However, it is a general principle that

> any initial value problem that involves only real numbers in the problem, should involve only real numbers in the answer, and customarily we rewrite the answer so that it only mentions real numbers

even though complex numbers may be convenient as an intermediate step in getting to that answer. How can we rewrite our answer so that it will not involve anything but real numbers?

In applications the complex numbers generally occur in pairs, so that we end up with terms like $Ae^{(P+Qi)t} + Be^{(P-Qi)t}$ where P and Q are some particular constants that we find, and A and B are arbitrary constants that we don't find (or that we find by plugging in some given initial conditions). (And even if we don't get a pair, we could *make* a pair, by taking one of A, B to be 0.) Keep in mind that A and B are *arbitrary* constants — they need not be real numbers. Now compute as follows:

$$\begin{aligned} Ae^{(P+Qi)t} + Be^{(P-Qi)t} &= e^{Pt}\left[Ae^{Qit} + Be^{-Qit}\right] \\ &= e^{Pt}\left[A(\cos Qt + i\sin Qt) + B(\cos Qt - i\sin Qt)\right] \\ &= e^{Pt}\left[(A+B)\cos Qt + (A-B)i\sin Qt\right] \\ &= e^{Pt}\left(\alpha\cos Qt + \beta\sin Qt\right) \end{aligned}$$

where α and β are new arbitrary constants. They are related to the old arbitrary constants:

$$\alpha = A + B, \qquad \beta = (A-B)i \qquad (*)$$

but in most cases we don't need to keep track of $(*)$, because we won't need to go back to the original A and B anyway. In most cases, all we need to know is that

> any function that can be expressed in the form $Ae^{(P+Qi)t} + Be^{(P-Qi)t}$ for some constants A and B, can also be expressed in the form $e^{Pt}\left(\alpha\cos Qt + \beta\sin Qt\right)$ for some constants α and β.

We can forget about the intermediate computation steps. And in applications, usually α and β turn out to be real numbers (though A and B might not be real).

By the way, if at some point you *do* need to go back to the exponential expressions, the transformation $(*)$ is reversible. To see that, just multiply both sides of the second equation by i; that yields

$$\alpha = A + B, \qquad i\beta = -A + B.$$

Now adding or subtracting, and dividing by 2, we obtain

$$A = \frac{\alpha - i\beta}{2}, \qquad B = \frac{\alpha + i\beta}{2}. \qquad (**)$$