Four people had perfect scores. The class average was 24.95 points out of 30, which is 83\(\frac{1}{6}\)%, a grade of B.

(7 points) Solve \(y'''' + 2y''' - 3y'' = 7\).

\textit{Solution.} Rewrite this as \(D^2(D - 1)(D + 3)y = 7\). The right side has annihilator \(D\). Hence the solution is of the form

\[y = a_1 + a_2x + p_1x^2 + a_3e^x + a_4e^{-3x}\]

where we must find \(p_1\). Temporarily replacing all the \(a\)'s with 0's,

\[
\begin{align*}
  y &= p_1x^2 \\
  y' &= 2p_1x \\
  y'' &= 0 \\
  y''' &= 2p_1 \\
  y''' &= 0 \\
  y'''' &= 0
\end{align*}
\]

\[
\begin{array}{c|c|c|c|c|c}
  y'''' & 2y''' & -3y'' & = & 0 & +0 & -6p_1 & = & 7 \\
\end{array}
\]

hence \(p_1 = -\frac{7}{6}\), and the answer is

\[
y = a_1 + a_2x - \frac{7}{6}x^2 + a_3e^x + a_4e^{-3x}.
\]

Partial credit: Deduct 2 points for each missing term or extraneous term; deduct 3 points for an error in the term that has the constant coefficient.

(7 points) Solve \(\frac{d^6 y}{dx^6} + 3\frac{d^5 y}{dx^5} + 3\frac{d^4 y}{dx^4} + \frac{d^3 y}{dx^3} = 3e^{-2x} + 2e^x\).

\textit{Solution.} Rewrite this as \(Ly = 3e^{-2x} + 2e^x\), where

\[L = D^6 + 3D^5 + 3D^4 + D^3 = (D + 1)^3D^3.\]

The right side of the problem, \(3e^{-2x} + 2e^x\), has annihilator \((D + 2)(D - 1)\). Thus the solution is of the form

\[y = (a_1 + a_2x + a_3x^2)e^{-x} + (a_4 + a_5x + a_6x^2) + p_1e^{-2x} + p_2e^x\]
where we need to find $p_1$ and $p_2$. Setting the $a$’s temporarily to 0,

\[
\begin{align*}
  y &= p_1 e^{-2x} + p_2 e^x \\
  y' &= -2p_1 e^{-2x} + p_2 e^x \\
  y'' &= 4p_1 e^{-2x} + p_2 e^x \\
  y''' &= -8p_1 e^{-2x} + p_2 e^x \\
  y^{(4)} &= 16p_1 e^{-2x} + p_2 e^x \\
  y^{(5)} &= -32p_1 e^{-2x} + p_2 e^x \\
  y^{(6)} &= 64p_1 e^{-2x} + p_2 e^x
\end{align*}
\]

which requires $p_1 = 3/8$ and $p_2 = 1/4$. Thus

\[
L_y = 8p_1 e^{-2x} + 8p_2 e^x \\
\]  

which requires $p_1 = 3/8$ and $p_2 = 1/4$. Thus

\[
y = (a_1 + a_2 x + a_3 x^2)e^{-x} + (a_4 + a_5 x + a_6 x^2) + \frac{3}{8} e^{-2x} + \frac{1}{4} e^x
\]

Partial credit: I deducted 1 point for most arithmetic errors, 2 points for mild conceptual errors, more points for bigger conceptual errors, and most of the points if a student was clueless.
Give the annihilator of lowest order for each of the following. You may leave the annihilator in factored form or multiply it out.

(2 points) $x^2 + 3x + 2e^{2x}$
Answer: $D^3(D - 2)$ or $D^4 - 2D^3$

(3 points) $xe^{7x} + \cos 5x$
Answer: $(D - 7)^2(D^2 + 25)$ or $(D - 7)(D - 5i)(D + 5i)$ or $(D^2 - 14D + 49)(D^2 + 25)$ or $D^4 - 14D^3 + 74D^2 - 350D + 1225$

(3 points) $xe^{7x}\cos 5x$
Answer: $(D - 7 - 5i)^2(D - 7 + 5i)^2$ or $(D^2 - 14D + 74)^2$ or $D^4 - 28D^3 + 344D^2 - 2072D + 5476$

(5 points) Solve $y'''' + 6y''' + 16y'' + 18y' + 7y = 0$.
Hint: That factors as $(D^2 + 4D + 7)(D + 1)^2y = 0$.
Solution. Factor it a little further, as $(D + 2 + \sqrt{3}i)(D + 2 - \sqrt{3}i)(D + 1)^2y = 0$. (Algebra not shown here.) Hence the general solution is
$$y = e^{-2x}(a \cos \sqrt{3}x + b \sin \sqrt{3}x) + ce^{-x} + dx e^{-x}.$$
(3 points) Find all the fifth roots of 5. Unlike in the homework problems, this time you may use sines and cosines to express your answer. (Moreover, you’ll probably need to. The answer can be expressed using square roots instead of sines and cosines, but that’s much harder and I don’t recommend trying it during the test.)

Solution. $\sqrt[5]{5} \left(\cos \theta + i \sin \theta\right)$, where $\theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$. The answer can also be expressed in other ways. For instance, other choices of five angles can be used, such as $0, \pm \frac{2\pi}{5}, \pm \frac{4\pi}{5}$. I will also give full credit for decimal answers:

$1.37972967, 0.42635999 \pm 1.3122009i, -1.1162247 \pm 0.8109847i$.

Extra for fanatics: The answer can be expressed in terms of square roots, though that is much harder. To see it, let’s look instead for the fifth roots of 1 (and then we can multiply them all by $\sqrt[5]{5}$). The problem is $x^5 - 1 = 0$. That factors as $(x-1)(x^4 + x^3 + x^2 + x + 1) = 0$. There is a formula for the general fourth degree polynomial, but it’s a horrible mess that I wouldn’t wish on anyone. However, the polynomial equation $x^4 + x^3 + x^2 + x + 1 = 0$ has a great deal of symmetry, so we can apply certain tricks to this polynomial — it is easier to solve than most fourth degree polynomials. Divide the equation through by $x^2$, so that we get

$$x^2 + x + 1 + x^{-1} + x^{-2} = 0.$$

Now, let $u = x + x^{-1}$. Then $u^2 = x^2 + 2 + x^{-2}$, and so our problem can be rewritten as

$$u^2 + u - 1 = 0, \quad \text{or} \quad (u + \frac{1}{2})^2 = \frac{5}{4}.$$

That’s a quadratic, which has two real roots, $u = -\frac{1}{2} \pm \frac{1}{2} \sqrt{5}$.

We’re going to have another plus-or-minus sign in the problem soon, and I don’t want to confuse the two. So let’s use $\sigma$ (Greek lower case letter sigma) for either a plus one or a minus one. (Think of s standing for “sign”.) Thus, we have $u = -\frac{1}{2} + \frac{\sigma}{2} \sqrt{5}$. Note that $\sigma^2 = 1$. Also note that $u$ is a solution of $u^2 + u - 1 = 0$, so $u^2 = 1 - u = \frac{3}{2} - \frac{\sigma}{2} \sqrt{5}$. In a few moments we will also need to work with the number $u^2 - 4 = -\frac{5 - \sigma \sqrt{5}}{2}$. Since $0 < \sqrt{5} < 5$, it follows that — regardless of whether $\sigma$ is positive or negative — the number $u^2 - 4$ is negative. We shall rewrite it as $u^2 - 4 = -\frac{5 + \sigma \sqrt{5}}{2}$. Now $u$ is a number, and we want to find $x$, the solution of the equation $x - u + x^{-1} = 0$. Multiply that equation through by $x$, and we get $x^2 - xu + 1 = 0$, another quadratic equation. The quadratic formula gives us the answer,

$$x = \frac{u \pm \sqrt{u^2 - 4}}{2} = \frac{u \pm \sqrt{-\frac{5 + \sigma \sqrt{5}}{2}}}{2} = \frac{u \pm i \sqrt{\frac{10 + 2\sigma \sqrt{5}}{4}}}{2}.$$
\[ -\frac{1}{2} + \frac{\sigma}{2} \sqrt{5} \pm i \frac{\sqrt{10 + 2\sigma \sqrt{5}}}{2} = 1 - \sigma \sqrt{5} + i \sqrt{10 + 2\sigma \sqrt{5}}. \]

Note that both \(\sigma\)'s are +1 or both \(\sigma\)'s are −1. Independently of \(\sigma\), the ± symbol is either +1 or −1. Thus there are four combinations, and four answers. We also get one answer from the factor of \((x - 1)\) which we divided out at the beginning of this explanation. Now multiply all the answers by \(\sqrt[5]{5}\), and we get these five answers to the problem:

\[
\sqrt[5]{5} \left( \frac{-1 + \sqrt{5}}{4} + i \frac{\sqrt{10 + 2\sqrt{5}}}{4} \right), \quad \sqrt[5]{5} \left( \frac{-1 - \sqrt{5}}{4} - i \frac{\sqrt{10 - 2\sqrt{5}}}{4} \right), \quad \sqrt[5]{5} \left( \frac{-1 + \sqrt{5}}{4} + i \frac{\sqrt{10 + 2\sqrt{5}}}{4} \right), \quad \sqrt[5]{5} \left( \frac{-1 - \sqrt{5}}{4} - i \frac{\sqrt{10 - 2\sqrt{5}}}{4} \right), \quad \sqrt[5]{5}.
\]

A byproduct of this computation is this interesting information: \(\cos \frac{2\pi}{5} = -\frac{1 + \sqrt{5}}{4}\) and \(\cos \frac{\pi}{5} = \frac{1 + \sqrt{5}}{4}\). (Also the sines, but they’re messier and thus less interesting.) The number \((\sqrt{5} - 1)/2\) is the reciprocal of the Golden Ratio, which shows up in pentagrams and other geometric figures that fascinated the ancient sect of the Pythagoreans; Leonardo Da Vinci used it in some of his artworks as well.