Math 198 Test 3, Thursday 31 March 2005, 4 pages, 30 points, 75 minutes.

Four people had perfect scores. The class average was 24.95 points out of 30, which is $83\frac{1}{6}\%$, a grade of B.

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(7 points) Solve $y'''' + 2y''' - 3y'' = 7$.

*Solution.* Rewrite this as $D^2(D-1)(D+3)y = 7$. The right side has annihilator $D$. Hence the solution is of the form

$$y = a_1 + a_2x + p_1x^2 + a_3e^x + a_4e^{-3x}$$

where we must find $p_1$. Temporarily replacing all the $a$'s with 0's,

$$\begin{array}{rcllll} y & = & p_1x^2 & & & \\ y' & = & & 2p_1x & & \\ y'' & = & & & 2p_1 & \\ y''' & = & 0 & & & \\ y'''' & = & 0 & & & \\ \hline y'''' + 2y''' - 3y'' & = & 0 & +0 & -6p_1 & \stackrel{?}{=} \ 7 \end{array}$$

hence $p_1 = -7/6$, and the answer is

$$\boxed{y = a_1 + a_2x - \frac{7}{6}x^2 + a_3e^x + a_4e^{-3x}}.$$

Partial credit: Deduct 2 points for each missing term or extraneous term; deduct 3 points for an error in the term that has the constant coefficient.

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(7 points) Solve $\dfrac{d^6y}{dx^6} + 3\dfrac{d^5y}{dx^5} + 3\dfrac{d^4y}{dx^4} + \dfrac{d^3y}{dx^3} = 3e^{-2x} + 2e^x$.

*Solution.* Rewrite this as $Ly = 3e^{-2x} + 2e^x$, where

$$L = D^6 + 3D^5 + 3D^4 + D^3 = (D+1)^3D^3.$$

The right side of the problem, $3e^{-2x} + 2e^x$, has annihilator $(D+2)(D-1)$. Thus the solution is of the form

$$y = (a_1 + a_2x + a_3x^2)e^{-x} + (a_4 + a_5x + a_6x^2) + p_1e^{-2x} + p_2e^x$$

where we need to find $p_1$ and $p_2$. Setting the $a$'s temporarily to 0,

$$\begin{array}{rcll|l} y & = & p_1e^{-2x} & +p_2e^x & \\ y' & = & -2p_1e^{-2x} & +p_2e^x & \\ y'' & = & 4p_1e^{-2x} & +p_2e^x & \\ y''' & = & -8p_1e^{-2x} & +p_2e^x & 1 \\ y^{(4)} & = & 16p_1e^{-2x} & +p_2e^x & 3 \\ y^{(5)} & = & -32p_1e^{-2x} & +p_2e^x & 3 \\ y^{(6)} & = & 64p_1e^{-2x} & +p_2e^x & 1 \\ \hline \\ [-10pt]Ly & = & 8p_1e^{-2x} & +8p_2e^x \\ & \stackrel{?}{=} & 3e^{-2x} & +2e^x \end{array}$$

which requires $p_1 = 3/8$ and $p_2 = 1/4$. Thus

$$\boxed{y = (a_1 + a_2x + a_3x^2)e^{-x} + (a_4 + a_5x + a_6x^2) + \frac{3}{8}e^{-2x} + \frac{1}{4}e^x}.$$

Partial credit: I deducted 1 point for most arithmetic errors, 2 points for mild conceptual errors, more points for bigger conceptual errors, and most of the points if a student was clueless.

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Give the annihilator of lowest order for each of the following. You may leave the annihilator in factored form *or* multiply it out.

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(2 points) $x^2 + 3x + 2e^{2x}$

*Answer*: $\boxed{D^3(D-2)}$ or $\boxed{D^4 - 2D^3}$

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(3 points) $xe^{7x} + \cos 5x$

*Answer:* $\boxed{(D-7)^2(D^2+25)}$ or $\boxed{(D-7)^2(D-5i)(D+5i)}$ or $\boxed{(D^2-14D+49)(D^2+25)}$ or $\boxed{D^4 - 14D^3 + 74D^2 - 350D + 1225}$

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(3 points) $xe^{7x} \cos 5x$

*Answer.* $\boxed{(D-7-5i)^2(D-7+5i)^2}$ or $\boxed{(D^2-14D+74)^2}$ or $\boxed{D^4 - 28D^3 + 344D^2 - 2072D + 5476}$

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(5 points) Solve $y'''' + 6y''' + 16y'' + 18y' + 7y = 0$.

*Hint*: That factors as $(D^2 + 4D + 7)(D+1)^2 y = 0$.

*Solution.* Factor it a little further, as $(D+2+\sqrt{3}i)(D+2-\sqrt{3}i)(D+1)^2 y = 0$. (Algebra not shown here.) Hence the general solution is

$$\boxed{y = e^{-2x}(a\cos\sqrt{3}x + b\sin\sqrt{3}x) + ce^{-x} + dxe^{-x}}.$$

*Partial credit.* Each of the following types of errors generally cost 1 point:

- Solving $k^2 + 4k + 7 = 0$ incorrectly (except if it changed the nature of the problem drastically — e.g., if you got two real solutions)
- Each computational error (e.g., replacing $e^{-2x}$ with $e^{2x}$)
- Each error in the form of the answer (e.g., writing $\sqrt{3}$ in place of $\sqrt{3}x$). (Two *different* errors of form cost two points.)
- Leaving the answer in a form that involved $i$. (I stated repeatedly in lectures that $i$ may be useful in intermediate steps, but a problem without $i$ should have an final answer without $i$.)

(3 points) Find all the fifth roots of 5. Unlike in the homework problems, **this time you may use sines and cosines** to express your answer. (Moreover, you'll probably *need* to. The answer *can* be expressed using square roots instead of sines and cosines, but that's much harder and I don't recommend trying it during the test.)

*Solution.* $\boxed{\sqrt[5]{5}\,(\cos\theta + i\sin\theta), \text{ where } \theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}}$. The answer can also be expressed in other ways. For instance, other choices of five angles can be used, such as $0, \pm\frac{2\pi}{5}, \pm\frac{4\pi}{5}$. I will also give full credit for decimal answers:

$$\boxed{1.37972967, \qquad 0.4263599 \pm 1.3122009i, \qquad -1.1162247 \pm 0.8109847i}.$$

*Extra for fanatics*: The answer *can* be expressed in terms of square roots, though that is much harder. To see it, let's look instead for the fifth roots of 1 (and then we can multiply them all by $\sqrt[5]{5}$). The problem is $x^5 - 1 = 0$. That factors as $(x-1)(x^4+x^3+x^2+x+1) = 0$. There is a formula for the general fourth degree polynomial, but it's a horrible mess that I wouldn't wish on anyone. However, the polynomial equation $x^4 + x^3 + x^2 + x + 1 = 0$ has a great deal of symmetry, so we can apply certain tricks to this polynomial — it is easier to solve than most fourth degree polynomials. Divide the equation through by $x^2$, so that we get

$$x^2 + x + 1 + x^{-1} + x^{-2} = 0.$$

Now, let $u = x + x^{-1}$. Then $u^2 = x^2 + 2 + x^{-2}$, and so our problem can be rewritten as

$$u^2 + u - 1 = 0, \qquad \text{or} \qquad (u + \frac{1}{2})^2 = \frac{5}{4}.$$

That's a quadratic, which has two real roots, $u = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$.

We're going to have another plus-or-minus sign in the problem soon, and I don't want to confuse the two. So let's use $\sigma$ (Greek lower case letter sigma) for either a plus one or a minus one. (Think of s standing for "sign".) Thus, we have $u = -\frac{1}{2} + \frac{\sigma}{2}\sqrt{5}$. Note that $\sigma^2 = 1$. Also note that $u$ is a solution of $u^2 + u - 1 = 0$, so $u^2 = 1 - u = \frac{3}{2} - \frac{\sigma}{2}\sqrt{5}$. In a few moments we will also need to work with the number $u^2 - 4 = \frac{-5-\sigma\sqrt{5}}{2}$. Since $0 < \sqrt{5} < 5$, it follows that — regardless of whether $\sigma$ is positive or negative — the number $u^2 - 4$ is negative. We shall rewrite it as $u^2 - 4 = -\frac{5+\sigma\sqrt{5}}{2} = -\frac{10+2\sigma\sqrt{5}}{4}$.

Now $u$ is a number, and we want to find $x$, the solution of the equation $x - u + x^{-1} = 0$. Multiply that equation through by $x$, and we get $x^2 - xu + 1 = 0$, another quadratic equation. The quadratic formula gives us the answer,

$$x = \frac{u \pm \sqrt{u^2 - 4}}{2} = \frac{u \pm \sqrt{-\frac{5+\sigma\sqrt{5}}{2}}}{2} = \frac{u \pm i\sqrt{\frac{10+2\sigma\sqrt{5}}{4}}}{2}$$

$$= \frac{-\frac{1}{2} + \frac{\sigma}{2}\sqrt{5} \pm \frac{i}{2}\sqrt{10 + 2\sigma\sqrt{5}}}{2} = \frac{-1 + \sigma\sqrt{5} \pm i\sqrt{10 + 2\sigma\sqrt{5}}}{4}.$$

Note that both $\sigma$'s are $+1$ or both $\sigma$'s are $-1$. Independently of $\sigma$, the $\pm$ symbol is either $+1$ or $-1$. Thus there are four combinations, and four answers. We also get one answer from the factor of $(x - 1)$ which we divided out at the beginning of this explanation. Now multiply all the answers by $\sqrt[5]{5}$, and we get these five answers to the problem:

$$\sqrt[5]{5}\left(\frac{-1+\sqrt{5}}{4} + i\frac{\sqrt{10+2\sqrt{5}}}{4}\right), \qquad \sqrt[5]{5}\left(\frac{-1-\sqrt{5}}{4} - i\frac{\sqrt{10-2\sqrt{5}}}{4}\right),$$

$$\sqrt[5]{5}\left(\frac{-1+\sqrt{5}}{4} + i\frac{\sqrt{10+2\sqrt{5}}}{4}\right), \qquad \sqrt[5]{5}\left(\frac{-1-\sqrt{5}}{4} - i\frac{\sqrt{10-2\sqrt{5}}}{4}\right), \qquad \sqrt[5]{5}.$$

A byproduct of this computation is this interesting information: $\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$ and $\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$. (Also the sines, but they're messier and thus less interesting.) The number $(\sqrt{5} - 1)/2$ is the reciprocal of the Golden Ratio, which shows up in pentagrams and other geometric figures that fascinated the ancient sect of the Pythagoreans; Leonardo Da Vinci used it in some of his artworks as well.