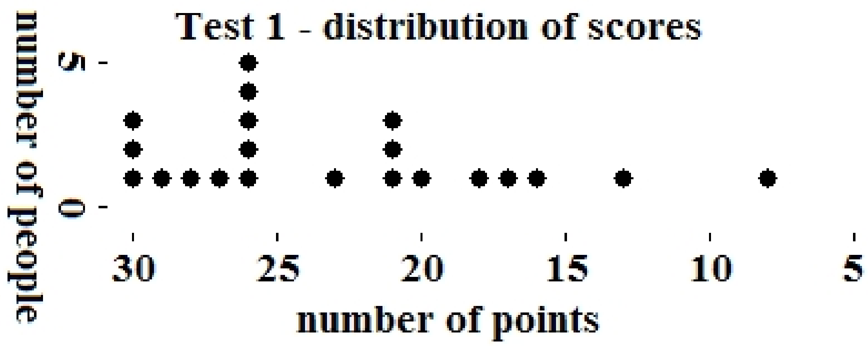
~~Friday 5 November 1999~~ Thursday 10 February 2005. (Most of the test is new, but I did copy the heading format from an old test, and I forgot to change the date.)

Math 198 Test 1, 30 points, 75 minutes. **ANSWERS AND COMMENTARY.**

There were three perfect scores. The full distribution of scores is shown below. The class average was 22.952 points out of 30, which surprised me — I expected higher scores. That's about 76.508%. The standard deviation (a measurement of how spread out the scores were) was 6.0206 points. I keep the numbers, not the letters, in my gradebook, but to give you an idea of how well you're doing, here is a tentative, approximate grading scale:

points	30	29	28	27	26	25	24	23	22	21	20	19	18	less than 18
percent	100			90			80			70			60	
letter	A	A	A	$A-$	$B+$	B	$B-$	$C+$	C	$C-$	$D+$	D	$D-$	F

(Sorry, there are no $A+$ grades in the College of Arts and Sciences.)



(7 points) $\dfrac{dy}{dx} = (x - y - 4)^2 + 2$. Solve explicitly for y; simplify as much as possible; circle your answer.

Solution. Use the **affine substitution** $u = x - y - 4$. (This is the *only* method that I've seen that works for this problem. Students who attempted other methods did not make any progress that I could give any credit for.) Then

$$\frac{du}{dx} \quad = \quad \frac{dx}{dx} - \frac{dy}{dx} - \frac{d4}{dx} \quad = \quad 1 - \left(u^2 + 2\right) - 0 \quad = \quad -u^2 - 1.$$

(Many students had the basic idea right but made an algebra error somewhere in the line above — i.e., as soon as they started the problem.) Now separate variables:

$$-dx = \frac{du}{u^2 + 1}.$$

Integrate both sides; $C - x = \arctan u$. Hence $u = \tan(C - x)$. Return to the original variables: $x - y - 4 = \tan(C - x)$. Solve for y: $\boxed{y = x - 4 - \tan(C - x)}$. Since tan is an odd function, we can rewrite that as $\boxed{y = x - 4 + \tan(x - C)}$; or, if you prefer, you can replace C with $-C$ (by choosing a different C).

Partial credit:

By one arithmetic error or another, several students arrived at $-dx = \frac{du}{u^2+b}$ for some number b other than 1. I gave 6 points if the remainder of the computation was carried out correctly from that point on, or less if further errors were made.

In case of $b > 0$, the remainder of the computation should have been as follows: Let $a = \sqrt{b}$. Use formula 22 in the integral table supplied with the test (which can also be obtained by methods from calculus). We get

$$-x = -\int dx = \int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + c_1$$

Rewrite that as $c_2 - ax = \arctan \frac{u}{a}$, or $u = a\tan(c - ax)$. That is, $x - y - 4 = \sqrt{b}\tan(c - \sqrt{b}x)$. Solve for y; thus $y = x - 4 + \sqrt{b}\tan(\sqrt{b}x - c)$, worth 6 points.

In case of $b < 0$, the remainder of the computation should have been as follows: Let $a = \sqrt{-b}$. Then we get

$$-x = -\int dx = \int \frac{du}{u^2 - a^2} = \frac{1}{2a}\int \left(\frac{1}{u-a} - \frac{1}{u+a}\right) du = \frac{1}{2a} \ln \left|\frac{u-a}{u+a}\right| + c_1.$$

Then $ce^{-2ax} = \frac{u-a}{u+a} = 1 - \frac{2a}{u+a}$, hence $\frac{2a}{u+a} = 1 - ce^{-2ax}$, hence $x - y - 4 + a = u + a = \frac{2a}{1-ce^{-2ax}}$, and finally $y = x - 4 + \sqrt{-b}\left(1 - \frac{2}{1-ce^{-2\sqrt{-b}x}}\right) = x - 4 - \sqrt{-b}\frac{1+ce^{-2\sqrt{-b}x}}{1-ce^{-2\sqrt{-b}x}}$. That, or anything equal to that, was worth 6 points; answers differing slightly from that were worth slightly less.

At any rate, $\int \frac{du}{u^2+b}$ definitely is *not* equal to $\ln|u^2 + b| + C$. That's on the common errors web page, and it was also discussed in class in our last review session before the test, so I was surprised to see this error on several papers, and I penalized 2 points for it. — If no further errors were made after *that*, the computation would be $-x = \ln|u^2 + b| + c_1$, hence $ce^{-x} = u^2 + b$, hence $x - y - 4 = u = \pm\sqrt{ce^{-x} + b}$, hence $y = x - 4 \pm \sqrt{ce^{-x} + b}$. For that answer (or for that answer with the $\pm$ replaced by either a plus sign or a minus sign), I gave 5 points in the case of $b = 1$, 4 points in the case of some other value of b, and fewer points for anything similar but not quite the same as this — i.e., if additional mistakes were made.

Some students believed that $\tan(x + C)$ is equal to $\tan(x) + \tan(C)$, or something like that. This is an instance of the "everything is additive" error in the *common errors* web page. It leads to an incorrect answer, and costs 2 points.

(8 points) $\dfrac{dy}{dx} = \dfrac{2y}{x} - x^2y^2$ with initial condition $y(2) = 1$. Solve explicitly for y; circle your answer.

Solution. Rewrite as

$$\frac{dy}{dx} - 2x^{-1}y = -x^2y^2.$$

This is a **Bernoulli** equation with $n = 2$. Let $u = y^{1-n} = y^{1-2} = y^{-1}$; then $y = u^{-1}$ and $\dfrac{dy}{dx} = -u^{-2}\dfrac{du}{dx}$. (Getting that far and then becoming clueless was worth 1 point.) Substituting those into the given differential equation yields

$$-u^{-2}\,\frac{du}{dx} - 2x^{-1}u^{-1} = -x^2u^{-2}.$$

Multiply through by $-u^2$ to obtain

$$\frac{du}{dx} + 2x^{-1}u = x^2$$

which is **linear** in standard form. A significant number of students made sign errors or other minor algebra errors by the time they got to this step; some of those errors are discussed below. But first, let's continue with the correct solution: The integrating factor is $e^{\int 2x^{-1}dx} = e^{2\ln|x|+c_1} = x^2$.

Now we have to multiply that onto *both* sides of the standard form linear equation, which is justified by the rule that if you do the same thing to two equal objects the results will be two equal objects. (Some students only multiplied the integrating factor onto *one* side of the standard form linear equation, a grievous conceptual error for which I charged 2 points.)

Multiplying through yields the **exact** differential equation

$$\frac{d}{dx}\left(x^2u\right) \quad \stackrel{?}{=} \quad x^2\frac{du}{dx} + 2xu \quad = \quad x^4.$$

(Actually, multiplying through yields the *second* of the two equations above. The quantity to the left of the questionmark is arrived at by another method; it consists of the derivative of the product of the integrating factor and the dependent variable. The questionmark indicates that we should check that the things to the left and right of the equal sign are indeed equal.)

Integrating both sides, $x^2u = \dfrac{1}{5}x^5 + C$. (You have to integrate *both* sides. Integrating only the left side is a grievous conceptual error, for which I charged 2 points this time.)

Multiplying both sides by x^{-2} yields $u = \dfrac{1}{5}x^3 + Cx^{-3}$. (It does *not* yield $u = \frac{1}{5}x^3 + C$. Common errors web page, loss of invisible parentheses. Penalty one point, this time.)

To find C, plug in the initial conditions: $x = 2$, $y = 1$, $u = 1$. Thus $C = -12/5$. We obtain

$$y^{-1} = u = \frac{1}{5}x^3 - \frac{12}{5}x^{-2} = \frac{x^5 - 12}{5x^2}$$

and finally

$$\boxed{y = \frac{5x^2}{x^5 - 12}}.$$

Here are a few of the **erroneous** sidetracks that people took:

- In place of $\frac{du}{dx}+\frac{2}{x}u=x^2$, if you get the equation $\frac{du}{dx}-\frac{2}{x}u=x^2$, then the integrating factor is x^{-2}. Thus, a mere sign error at this step yields an enormous alteration in the problem; consequently I'm charging two points for this error, If you make no further errors after that, here's the computation: Multiplying through yields

$$\frac{d}{dx}\left(x^{-2}u\right)=x^{-2}\frac{du}{dx}-2x^{-3}u=1.$$

  Integrate both sides; thus $x^{-2}u=x+c$, so $y^{-1}=u=x^3+cx^2$, To find c, plug in $(x,y)=(2,1)$; thus $1=8+4c$, so $c=-7/4$. The answer, then, is $y=1/(x^3-(7x^2/4))$, or $y=4/(4x^3-7x^2)$. I gave 6 points for that answer, or slightly less for answers differing slightly from this.

- In place of $\frac{du}{dx}+\frac{2}{x}u=x^2$, if you get the equation $\frac{du}{dx}+\frac{1}{2x}u=x^2$, then the remaining steps should be: Integrating factor is $x^{1/2}$; multiplying through yields

$$\frac{d}{dx}\left(x^{1/2}u\right)=x^{1/2}\frac{du}{dx}+\frac{1}{2}x^{-1/2}u=x^{5/2}.$$

  Integrating both sides yields $x^{1/2}u=\frac{2}{7}x^{7/2}+c_1$; then $y^{-1}=u=\frac{2}{7}x^3+c_1x^{-1/2}$; etc.

- When you make that first substitution into a Bernoulli equation, it should immediately yield a linear equation — i.e., an equation of the form $\frac{du}{dx}+P(x)u=f(x)$. Note that the term after $P(x)$ is u raised to the power 1, not to any other power, and the right side of the equation does not involve any u at all. If you get something of a different form, such as

$$\frac{du}{dx}+\frac{2}{x}u^{-3}=x^2 \qquad \text{or} \qquad \frac{du}{dx}+\frac{2}{x}u=x^2u,$$

  then you should immediately realize that you've made an algebra error somewhere, and you should not proceed further until you find and fix the error. Getting this far and then becoming clueless was worth 3 points. And by the way, $\int x^2u\partial x$ might be equal to $\frac{1}{3}x^3u+c$ in some contexts, but $\int x^2udx$ is *not* equal to $\frac{1}{3}x^3u+c$ when u is not a constant.

(7 points) Find the differential equation whose general solution is the two-parameter family of curves

$$y=ax^2+b.$$

(Those are the parabolas that are symmetric about the y-axis.) Simplify, and circle your answer.

Solution. Differentiating once with respect to x yields $y' = 2ax$; getting that far was worth 2 points. Now divide by x; thus $x^{-1}y' = 2a$; getting that far was worth 4 points. Differentiate again with respect to x; thus $-x^{-2}y' + x^{-1}y'' = 0$; that's worth 6 points.

Finally, the instructions did say *simplify*; to get the full 7 points you had to cancel out at least one factor of x. I preferred the answer of $\boxed{xy'' = y'}$ but also gave full credit for some other, comparable answers, such as $y'' = x^{-1}y'$ or $y'' - x^{-1}y' = 0$, etc.

(8 points) $(12xy^3 - 5x^4)\dfrac{dy}{dx} = 20x^3y - 3y^4$. Answer may be left in implicit form, but simplify as much as possible and circle your answer.

Solution. Nearly everyone got this problem right, generally by finding that the equation is **exact**: If we take $M = 20x^3y - 3y^4$ and $N = 5x^4 - 12xy^3$, we discover that $\partial M/\partial y = 20x^3 - 12y^3 = \partial N/\partial x$. Thus the problem is exact. We seek a function f that satisfies simultaneously **both** of these two conditions:

$$\begin{array}{rcc|rcc} M & = & \partial f/\partial x & N & = & \partial f/\partial y \\ f & = & \int M\partial x & f & = & \int N\partial y \\ & = & \int(20x^3y - 3y^4)\partial x & & = & \int(5x^4 - 12xy^3)\partial y \\ & = & 5x^4y - 3xy^4 + c_1(y) & & = & 5x^4y - 3xy^4 + c_2(x) \end{array}$$

Thus we may take $c_1(y)$ and $c_2(x)$ both equal to a constant c_3, and the general solution of the differential equation is $\boxed{5x^4y - 3xy^4 = C}$.

Alternate method. This method is (in my opinion) more obvious, but it also (at least in this example) requires more work. Since both $12xy^3 - 5x^4$ and $20x^3y - 3y^4$ are polynomials that are homogeneous of degree 4, it should be evident at a glance that this problem is **homogeneous**. Rewrite it as

$$(12xy^3 - 5x^4)dy = (20x^3y - 3y^4)dx.$$

This can be solved by *either* of two substitutions:

$y = ux, \quad dy = udx + xdu$	$x = vy, \quad dx = vdy + ydv$
$(12x^4u^3 - 5x^4)(udx + dxu)$ $= (20x^4u - 3u^4x^4)dx$	$(12vy^4 - 5v^4y^4)dy$ $= (20v^3y^4 - 3y^4)(vdy + ydv)$
Divide out x^4	Divide out y^4
$(12u^3 - 5)(udx + xdu) = (20u - 3u^4)dx$	$(12v - 5v^4)dy = (20v^3 - 3)(vdy + ydv)$
Move dx terms to right	Move dy terms to left
$(12u^3 - 5)xdu =$ $(20u - 3u^4 - 12u^4 + 5u)dx$	$(12v - 5v^4 - 20v^4 + 3v)dy$ $= (20v^3 - 3)ydv$
Simplify	Simplify
$(12u^3 - 5)xdu = 5(-3u^4 + 5u)dx$	$5(-5v^4 + 3v)dy = (20v^3 - 3)ydv$

Separate variables	**Separate variables**
$-5\dfrac{dx}{x} = \dfrac{12u^3 - 5}{3u^4 - 5u}du$	$-5\dfrac{dy}{y} = \dfrac{20v^3 - 3}{5v^4 - 3v}dv$
Rewrite for integrating (tricky?)	Rewrite for integrating (tricky?)
$-5\dfrac{dx}{x} = \dfrac{d(3u^4 - 5u)}{3u^4 - 5u}$	$-5\dfrac{dy}{y} = \dfrac{d(5v^4 - 3v)}{5v^4 - 3v}$
Integrate both sides	Integrate both sides
$-5\ln\lvert x\rvert = \ln\lvert 3u^4 - 5u\rvert - C_1$	$-5\ln\lvert y\rvert = \ln\lvert 5v^4 - 3v\rvert - C_1$
$C_2 = \ln\lvert(3u^4 - 5u)x^5\rvert$	$C_2 = \ln\lvert(5v^4 - 3v)y^5\rvert$
$C_3 = (3u^4 - 5u)x^5$	$C_3 = -(5v^4 - 3v)y^5$
plug in $u = y/x$	plug in $v = x/y$

and either way we get $\boxed{3xy^4 - 5x^4y = C}$. The so-called "tricky" step could instead be handled by partial fractions; perhaps that is more obvious but it is more work too.