Math 198 Final exam, Spring 2005, 6 pages, 50 points, 120 minutes.

The high score was 46. The class average was 27.35%. The scores were lower than I expected, and also they were spread out quite a bit more than I expected. In fact, it is almost as though I taught two classes: Students with scores under 20 (generally students who did not know algebra or calculus) and students with scores over 31.

All differential equations must be SOLVED EXPLICITLY FOR $y$.

(1 points) What is the power series for $e^x$?

Answer:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

(1 points) What is the power series for $e^{x^2}$? (Write enough terms to make the pattern evident.)

Answer: I think this problem was fairly easy if you attended the last few classes of the semester, but apparently rather difficult otherwise. The easiest method is to replace $x$ with $x^2$ in the previous problem, to obtain

$$e^{x^2} = 1 + (x^2) + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \frac{(x^2)^4}{4!} + \cdots$$

which is already acceptable for full credit, though I would have preferred that you rewrite it as

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots$$

A much harder method that also yields the right answer is to figure $f(x) = e^{x^2}$, hence $f'(x) = 2xe^{x^2}$ and $f''(x) = (4x^2 + 2)e^{x^2}$, and so on; hence $f(0) = 1$, $f'(0) = 0$, $f''(0) = 2$, and so on, hence

$$e^{x^2} = 1 + 0x + \frac{2x^2}{2!} + 0x^3 + \frac{12x^4}{4!} + 0x^5 + \frac{120x^6}{6!} + 0x^7 + \frac{1680x^8}{8!} + 0x^9 + \cdots$$

(4 points) Find the radius of convergence of the series $1 - \frac{x^3}{2} + \frac{x^6}{4} - \frac{x^9}{8} + \frac{x^{12}}{16} - \frac{x^{15}}{32} + \cdots$.

Answer: Some students, ignoring the content of my last few lectures, attempted to apply the ratio test to the coefficients

$$c_0 = 1, \quad c_1 = c_2 = 0, \quad c_3 = -\frac{1}{2}, \quad c_4 = c_5 = 0, \quad c_6 = \frac{1}{4}, \quad c_7 = c_8 = 0, \quad c_9 = -\frac{1}{8}, \ldots$$
and thus attempted to find a limit for the sequence of ratios
\[
\frac{c_0}{c_1} = \frac{1}{0} = ?, \quad \frac{c_1}{c_2} = \frac{0}{0} = ?, \quad \frac{c_2}{c_3} = \frac{0}{-\frac{1}{2}} = 0, \quad \frac{c_3}{c_4} = \frac{-\frac{1}{2}}{0} = ?, \quad \frac{c_4}{c_5} = \frac{0}{0} = ?, \quad \frac{c_5}{c_6} = \frac{0}{\frac{1}{4}} = 0, \ldots
\]
which is useless.

The correct procedure, which I discussed in class, was to substitute \( u = x^3 \); this converts the given series to
\[
1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \frac{u^4}{16} - \frac{u^5}{32} + \cdots = \sum_{j=0}^{\infty} \left( \frac{-u}{2} \right)^j.
\]
That’s the geometric series, which converges to \( \frac{1}{1 - (-u/2)} = 2/(2 + u) \) when \( \left| \frac{-u}{2} \right| < 1 \); that is, when \( |u| < 2 \). Or, if you didn’t happen to recognize the geometric series, you could use the ratio test: We have \( c_n = (-1/2)^n \), so \( |c_n/c_{n+1}| = 2 \), and so the series with the \( u \)'s has radius of convergence equal to \( \lim_{n \to \infty} |c_n/c_{n+1}| = 2 \).

At this point some students stopped with the answer of 2. I gave half credit for that answer.

The series with the \( u \)'s converges when \( |u| < 2 \). Hence the original series converges when \( |x^3| < 2 \); that is, when \( |x| < \frac{\sqrt[3]{2}}{2} \). So the radius of convergence for the original series is \( R = \frac{3\sqrt[3]{2}}{2} \).

(8 points) \( xy'' + \left( \frac{2}{3} - x \right)y' - y = 0 \). (The next page is blank, for further work on this or other problems.)

Answer: Problems like this one are a real pain to grade. Over half the class got most of the problem right, and deserve a substantial amount of partial credit. Most made a few minor errors. But most made different errors, and it takes a long time to figure out each paper. Ouch. Still, it can’t be helped; you do have to be tested on this stuff.

\[
\begin{align*}
y &= \sum_{n=0}^{\infty} c_n x^{n+r} & -1 \\
y' &= \sum_{n=0}^{\infty} (n + r) c_n x^{n+r-1} \\
 &= \sum_{n=-1}^{\infty} (n + r + 1) c_{n+1} x^{n+r} & 2/3 \\
x y' &= \sum_{n=0}^{\infty} (n + r) c_n x^{n+r} & -1 \\
y'' &= \sum_{n=0}^{\infty} (n + r - 1)(n + r) c_n x^{n+r-2} \\
x y'' &= \sum_{n=0}^{\infty} (n + r - 1)(n + r) c_n x^{n+r-1} \\
 &= \sum_{n=-1}^{\infty} (n + r)(n + r + 1) c_{n+1} x^{n+r} & 1 \\
xy'' + \left( \frac{2}{3} - x \right)y' - y &= \sum_{n=-1}^{\infty} \left\{ -c_n + \frac{2}{3}(n + r + 1) c_{n+1} - (n + r) c_n \\
 + (n + r)(n + r + 1) c_{n+1} \right\} x^{n+r}
\end{align*}
\]
hence
\[
\left\{ -c_n + \frac{2}{3}(n + r + 1) c_{n+1} - (n + r) c_n + (n + r)(n + r + 1) c_{n+1} \right\} = 0 \quad (n \geq -1)
\]
which simplifies to
\[
\left(\frac{2}{3} + n + r\right) (n + r + 1)c_{n+1} = (n + r + 1)c_n \quad (n = -1, 0, 1, 2, 3, \ldots)
\]
Since \(c_{-1} = 0\) and \(c_0 \neq 0\), when \(n = -1\) the equation above simplifies to \((r - \frac{1}{3})r = 0\), hence the roots are 0 and 1/3.

For either of those values of \(r\), and for all of \(n = 0, 1, 2, 3, \ldots\), the factor \((n + r + 1)\) is never 0, so we can divide it out. (That’s not just to make our work prettier, but to avoid a tremendous amount of labor and error-risk in the subsequent steps.) That leaves \((\frac{2}{3} + n + r)c_{n+1} = c_n\), which simplifies to
\[
c_{n+1} = \frac{c_n}{\frac{2}{3} + n + r} \quad (n = 0, 1, 2, 3, \ldots)
\]
**Partial credit:** 3 points for getting the information in either of the following two boxes, or 4 points for getting the information in both boxes.

<table>
<thead>
<tr>
<th>(r = 0) or (r = 1/3)</th>
<th>(c_{n+1} = \frac{c_n}{\frac{2}{3} + n + r}) ((n = 0, 1, 2, 3, \ldots))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(r = 0, c_{n+1} = \frac{3c_n}{3n + 2})</td>
<td>(r = 1/3, c_{n+1} = \frac{c_n}{\frac{2}{3} + n + r})</td>
</tr>
</tbody>
</table>

Now continue. For \(r = 1/3\), we get \(c_n = \frac{c_0}{n!}\). For \(r = 0\), we get
\[
\begin{align*}
c_0 & = \text{arbitrary} \\
(n = 0) & c_1 = 3c_0/2 = 3c_0/2 \\
(n = 1) & c_2 = 3c_1/5 = 3^2c_0/(2 \cdot 5) \\
(n = 2) & c_3 = 3c_2/8 = 3^3c_0/(2 \cdot 5 \cdot 8) \\
(n = 3) & c_4 = 3c_3/11 = 3^4c_0/(2 \cdot 5 \cdot 8 \cdot 11) \\
(n = 4) & c_5 = 3c_4/14 = 3^5c_0/(2 \cdot 5 \cdot 8 \cdot 11 \cdot 14)
\end{align*}
\]
and so on.

A common error. Note the leftmost column, which I have noted with the word “note.” I included that column in my examples on the chalkboard, but apparently I did not stress its usefulness enough. Some students who did not write that column, made the following type of error in their substitutions. Use the formula \(c_{n+1} = 3c_n/(3n + 2)\). To calculate \(c_1\), some students substituted \(n = 1\), and so they decided \(c_1 = 3c_0/(3 \cdot 1 + 2) = 3c_0/5\); similarly substituting \(n = 2\) gave them \(c_2 = 3c_1/(3 \cdot 2 + 2) = 3c_1/8\), and so on. But that’s not right, and I deducted a point for the answers that resulted from this type of error (if no other errors were made). The correct substitutions, of course, are: \(n = 0\) gives us \(c_1 = 3c_0/(3 \cdot 0 + 2) = 3c_0/2\), and \(n = 1\) gives us \(c_2 = 3c_1/(3 \cdot 1 + 2) = 3c_1/5\).

Another common error. Several students neglected to carry out the computation in the column that I have marked “second note.” Instead they made the mistake of plugging in \(c_0\) for \(c_n\) — thus obtaining \(c_2 = 3c_0/5\), \(c_3 = 3c_0/8\), \(c_4 = 3c_0/11\), etc. This type of error is partly conceptual, so I charged 2 points for it, in those cases when I could identify it — which usually means when it was not combined with other errors.

**Partial credit.** If you got all the main ideas up to this point, then:
• getting one of the two series in the answer completely right and the other series in the answer mostly wrong, was worth 6 points.

• getting one of the two series right and the other mostly right, was worth 7 points.

Finally, the solution to the problem is

\[
y = Ax^{1/3} \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \right) \\
+ B \left( 1 + \frac{3x}{2} + \frac{3^2 x^2}{2 \cdot 5} + \frac{3^3 x^3}{2 \cdot 5 \cdot 8} + \frac{3^4 x^4}{2 \cdot 5 \cdot 8 \cdot 11} + \cdots \right)
\]

or

\[
y = Ax^{1/3} e^x + B \left( 1 + \frac{3x}{2} + \frac{3^2 x^2}{2 \cdot 5} + \frac{3^3 x^3}{2 \cdot 5 \cdot 8} + \frac{3^4 x^4}{2 \cdot 5 \cdot 8 \cdot 11} + \cdots \right)
\]

A common error. Because \(c_0\) is arbitrary, the chart for computing the \(c_j\)’s sometimes is written with the \(c_0\) row omitted. Consequently, some students forgot that term in their answers — i.e., they omitted the “1 +” at the left end of the expression inside the big parentheses. I deducted 1 point for this type of error.

Partial checking: One of the two linearly independent solutions is fairly easy to check.

If \(y = x^{1/3} e^x\)
then \(y' = \frac{1}{3} x^{-2/3} e^x\)
\(y'' = \frac{2}{3} x^{-5/3} e^x\)
\(xy' = \frac{1}{3} x^{1/3} e^x\)
\(xy'' = \frac{2}{9} x^{-2/3} e^x\)

\[
x y'' + \left( \frac{2}{3} - x \right) y' - y = 0 \quad \sqrt{\text{(7 points)}}
\]

Answer:

Rewrite as \(\frac{(D - 1)x}{Dx} + 2Dy = t\), \(\frac{-y}{-y} = 1\). Solve the second equation for \(y\); thus \(y = Dx - 1\). Plug that into the first equation; thus the first equation becomes \((D - 1)x + 2D(Dx - 1) = t\). (Keep in mind that \(D1 = 0\).) That equation simplifies to which simplifies to \((2D^2 + D - 1)x = t\) or \(2(D - \frac{1}{2})(D + 1) = t\), which has general solution \(x(t) = Ae^{t/2} + Be^{-t} - t - 1\).

Then compute \(y = Dx - 1\), so \(y = \frac{1}{2} a e^{t/2} - be^{-t} - 2\). Alternate formulation: To eliminate a fraction in the answer, substitute \(A = \frac{1}{2} a\) and \(B = b\). Thus \(x(t) = 2Ae^{t/2} + Be^{-t} - t - 1\) and \(y(t) = Ae^{t/2} - be^{-t} - 2\).
Checking:

\[
\begin{array}{c|cccc}
      & x = 2Ae^{t/2} + Be^{-t} -t -1 & y = Ae^{t/2} - Be^{-t} -2 & x' = Ae^{t/2} - Be^{-t} -1 & y' = \frac{1}{2} Ae^{t/2} + Be^{-t} \\
\hline
      & -1 & -1 & 1 & 1
\end{array}
\]

\[
x' + x + 2y' = t
\]

\[
x' - y = 1
\]

Common errors: I did not deduct any points for writing \(a/2\) as \(1/2a\), though I should have. I did not deuct any points for replacing all the \(t\)’s in the answer with \(x\)’s, though I should have. Only 6 students got full credit on this one. I gave 5 points for any answer of the form

\[
x(t) = 2Ae^{t/2} + Be^{-t} + \left( \begin{array}{c}
\text{some polynomial in } t \\
\text{of degree } \leq 1
\end{array} \right)
\]

\[
y(t) = Ae^{t/2} - Be^{-t} + \left( \begin{array}{c}
\text{some other polynomial in } t \\
\text{of degree } \leq 1
\end{array} \right)
\]

I gave 3 points for any answer of the form

\[
x(t) = Ae^{t/2} + Be^{-t} + \text{(some function of } t) \\
y(t) = Ae^{t/2} + Be^{-t} + \text{(some other function of } t)
\]

I gave 1 point for any answer that included both \(x(t) = \text{something}\) and \(y(t) = \text{something}\).

Answer:

This is a constant-coefficient equation, so the solutions are of the form \(y = e^{kt}\). The number \(k\) is the solution of \(k^3 + 1 = 0\). That is, the values of \(k\) are the cube roots of \(-1\). One of those values is \(-1\). The other two values can be found by geometry — the numbers \(-1\) and the other two values are equally spaced along a circle centered at the origin. Or, this may be simpler: Divide \(k^3 + 1\) by \(k + 1\) (e.g., using synthetic division); we find \(k^3 + 1 = (k + 1)(k^2 - k + 1)\). So the remaining two roots are the solutions of the quadratic equation \(k^2 - k + 1\). Use the quadratic formula; we get \(k = \frac{1}{2} \pm \frac{1}{2} i \sqrt{3}\). Thus the answer is

\[
y = Ae^{-x} + e^{x/2} \left[ B \cos\left(\frac{1}{2} \sqrt{3} x\right) + C \sin\left(\frac{1}{2} \sqrt{3} x\right) \right].
\]

I did not deduct any points for writing \(1/2x\) instead of \(x/2\), though probably I should have. Next year I’ll make a bigger fuss about that.

About a third of the class got full credit on this one.

I deducted 1 point from students who clearly had the right idea but made some simple error of arithmetic, algebra, or transcription. One of the most common one-point errors was in writing \(e^{1/2}\) instead of \(e^{x/2}\).

I deducted 2 points from students whose solutions to the third degree polynomial differed in substantial ways from the correct solutions. I also deducted 2 points for an answer of \(y = \)
A \exp(-x) + B \exp \left( (\frac{1}{2} + \frac{i}{2}\sqrt{3})x \right) + C \exp \left( (\frac{1}{2} - \frac{i}{2}\sqrt{3})x \right), \text{ because (as I repeatedly specified in class) the number } i \text{ should only appear in intermediate steps, not in your answer.}

(6 points) \quad \frac{dy}{dx} = (y + 2x - 2)^2 - 1 \text{ with } y(1) = 1.

Answer:

Use the affine substitution \( u = y + 2x - 2 \). Then \( u' = y' + 2 = ((y + 2x - 2)^2 - 1) + 2 = u^2 + 1 \). That is, \( \frac{du}{dx} = u^2 + 1 \), so \( x = \int dx = \int (u^2 + 1)\, du = \arctan u + c = \arctan(y + 2x - 2) + c \), or \( \tan(x - c) = y + 2x - 2 \). To find \( c \), plug in \( x = y = 1 \); that yields \( \tan(1 - c) = 1 \); hence \( 1 - c = \pi/4 \), and \( -c = (\pi/4) - 1 \). Thus \( \tan(x + \frac{\pi}{4} - 1) = y + 2x - 2 \), or \( y = -2x + 2 + \tan \left( x + \frac{\pi}{4} - 1 \right) \). I also gave full credit for \( y = -2x + 2 + \tan \left( x - 0.2146 \right) \).

To check: Recall that \( \tan^2 \theta + 1 = \sec^2 \theta \). If \( y = -2x + 2 + \tan(x + \frac{\pi}{4} - 1) \), then \( \frac{dy}{dx} = -2 + \sec^2(x + \frac{\pi}{4} - 1) = -2 + 1 + \tan^2(x + \frac{\pi}{4} - 1) = -1 + (y + 2x - 2)^2 \).

Penalty 1 point for not finding \( y \) explicitly (see instructions on page 1 of exam). Penalty of 2 points for not finding \( c \).

Some of the errors were a bit strange. Students who take a course in differential equations should first have mastered a few basics of calculus; I mentioned that on the first day of the semester. Anyone who takes differential equations without knowing the chain rule deserves to flunk for that alone. Nevertheless, in future semesters I guess I will spend a few more minutes on the chain rule at the beginning of the semester.

(6 points) \quad \text{Given the solution } y = Ax + B(x^2 + 1), \text{ find the differential equation.}

Answer:
\[
y = Ax + B(x^2 + 1)
\]

isolate \(A\)
\[
x^{-1}y = A + B(x + x^{-1})
\]

differentiate both sides
\[
x^{-1}y' - x^{-2}y = B(1 - x^{-2})
\]
simplify: multiply through by \(x^2\)
\[
xy' - y = B(x^2 - 1)
\]
isolate \(B\)
\[
\frac{xy' - y}{x^2 - 1} = B
\]
differentiate both sides
\[
\frac{(xy' - y)'(x^2 - 1) - (xy' - y)(x^2 - 1)'}{(x^2 - 1)^2} = 0
\]
multiply through by denominator
\[
(xy' - y)'(x^2 - 1) - (xy' - y)(x^2 - 1)' = 0
\]
add same to both sides
\[
(xy' - y)'(x^2 - 1) = (xy' - y)(x^2 - 1)'
\]
rearrange terms
\[
(xy'')(x^2 - 1) = 2x(xy' - y)
\]
\[
(x^2 - 1)y'' = 2(xy' - y).
\]

Or, compute the functions in the left column; then do some algebraic elimination to discover what numbers in the right column will make everything cancel out.

<table>
<thead>
<tr>
<th>(y)</th>
<th>(y')</th>
<th>(y'')</th>
<th>(x^2y'')</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Ax) + (Bx^2) + (B)</td>
<td>(A) + (2Bx)</td>
<td>(2B)</td>
<td>(2Bx^2)</td>
</tr>
</tbody>
</table>

\[-2y + y'' + 2xy' - x^2y = 0Ax + 0Bx^2 + 0B\]

Apparently I overestimated the class; only two students got full credit on this problem. I couldn’t make much sense out of the wrong answers on this problem, so I didn’t give out much partial credit. I did give one point for these answers, since they are at least true statements (though they are not the answer to the problem): \(y''' = 0\) or \(y'' = 2B\) or \(y' = A + 2Bx\).

\[\frac{dy}{dx} = \frac{2y - x}{x}.\]

\textbf{Answer:}

This problem is homogeneous. Hence we can substitute \(y = ux\) and \(dy = u\,dx + x\,du\). (If you messed up that equation, then you haven’t understood the Chain Rule and Product Rule from calculus.) Simplifying and separating variables yields \(\frac{du}{u - 1} = \frac{dx}{x}\). Integrate both sides; thus \(\ln|u - 1| = c_1 + \ln|x|\). Raising \(e\) to the power on both sides yields \(u - 1 = cx\). (Note that the
right side should be \( cx \), not \( c + x \). That’s because \( e^{a+b} \) is equal to \( e^a e^b \), not \( e^a + e^b \). I charged 3 points for an error in this step.) That is, \( u = cx + 1 \), so \( y = cx^2 + x \).

A second method: The equation is linear. Rewrite it as \( y' - \frac{2}{x} y = -1 \). The integrating factor to use is \( e^{\int \frac{2}{x} dx} = x^2 \). Multiplying the differential equation through by that amount yields \((x^2y)' = x^{-2}y' - 2x^{-3}y = -x^{-2} \). Integrate both sides; thus \( x^{-2}y = x^{-1} + c \). Now multiply both sides by \( x^2 \), to arrive at the answer. A common error in this procedure was to misplace the \( c \), or omit it altogether; I charged 3 points for that.

Another error made by several students was to use an integrating factor of \( e^{\int \frac{2}{x} dx} = x^2 \). When you multiply the differential equation through by that amount, you get \( x^2 y' - 2xy = -x^2 \). The students then wrote the next step to be \((x^2y)' = -x^2 \), not bothering to check whether \( x^2y' - 2xy \) is in fact equal to \((x^2y)' \). It’s not. This is a checking step that I mentioned repeatedly in class. I charged 3 points for this error.

(6 points) \( xy' + 2y = 3x^3y^{-2} \).

\textbf{Answer:}

Rewrite this equation in standard form as \( y' + 2x^{-1}y = 3x^2y^{-2} \). This is a Bernoulli equation with \( n = -2 \); observing that fact was worth 1 point. Calculate

\[ v = y^{1-n} = y^3 \quad \text{and} \quad v' = 3y^2y' \; ; \]

getting all of that correctly was worth 2 points. (Some students who still do not understand the chain rule from calculus did not get this far.)

Multiply the standard form equation through by \( 3y^2 \); obtain \( 3y^2y' + 6x^{-1}y^3 = 9x^2 \). That is,

\[ v' + 6x^{-1}v = 9x^2, \]

worth 3 points. This is linear first-order, in standard form. The integrating factor is \( I(x) = e^{\int 6x^{-1} dx} = e^{6 \ln x} = x^6 \). Multiply through by that factor; we obtain \( x^6v' + 6x^5v = 9x^8 \). That is,

\[ \frac{d}{dx} \left( x^6v \right) = \frac{d}{dx} \left( x^9 \right) . \]

As I repeatedly explained in class, at this point you should check that \( \frac{d}{dx} \left( x^6v \right) \) (or whatever you have in its place) is actually equal to \( x^6v' + 6x^5v \) (or whatever you have in its place); checking this takes very little time and would have helped many students catch their errors.

Integrate both sides; \( x^6v = x^9 + c \). Divide out \( x^6 \); thus \( y^3 = v = x^3 + cx^{-6} \). Solve for \( y \); thus \( y = \frac{3}{\sqrt[3]{x^3 + cx^{-6}}} \). That last function is \textit{not} equal to \( \frac{3}{\sqrt[3]{x^3}} + \frac{3}{\sqrt[3]{cx^{-6}}} = x + c_1x^{-2} \), see the “common errors” web page.

Only 4 students got full credit for this problem. I gave 5 points out of 6 for answers of the form \( y = \frac{3}{\sqrt[3]{kx^3 + cx^{-6}}} \) where \( k \) was some particular number other than 1.