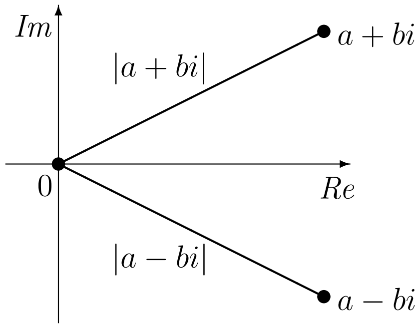
Math 198 Test 2, Tuesday 1 March 2005, 4 pages, 30 points, 75 minutes.

One student had a perfect score. The class average was 24.38 points out of 30, which is about 81.3%, or a grade of B-minus.

(6 points) Solve $x^2 + 6x + 11 = 0$. Also, find the absolute values of the two answers.



Solution. Completing the square: $x^2+6x+9 = -2$, so $(x+3)^2 = (\sqrt{2}i)^2$, and $x = \boxed{-3 \pm \sqrt{2}i}$. Also, $|-3 \pm \sqrt{2}i| = \sqrt{3^2 + (\sqrt{2})^2} = \sqrt{9+2} = \boxed{\sqrt{11}}$.

Common errors: Several students arrived at $\sqrt{9-2} = \sqrt{7}$ for their absolute value, or for one of their absolute values. I deducted one point if $\sqrt{11}$ and one other number were given as the two absolute values; I deducted two points if two numbers (the same or different from each other) different from $\sqrt{11}$ were given as absolute values. I did not deduct any points if $\sqrt{11}$ was given just once, and no other number was given, as the absolute value. You should have known that $|a + bi| = |a - bi|$, by symmetry. See diagram at right; the graphical symmetry between $a + bi$ and its complex conjugate $a - bi$ was mentioned in class.

(6 points) Find the general solution of $x^2y'' - 3xy' - y = 0$.

Solution. This is a Cauchy-Euler equation. The associated polynomial is $k(k-1) - 3k - 1 = 0$, which simplifies to $k^2 - 4k - 1 = 0$ or $k^2 - 4k + 4 = 5$. Hence $k = 2 \pm \sqrt{5}$. Thus the answer is $\boxed{y = ax^{2+\sqrt{5}} + bx^{2-\sqrt{5}}}$ or $\boxed{y = (ax^{\sqrt{5}} + bx^{-\sqrt{5}})x^2}$.

(10 points) Find the general solution of $(x^2 + \frac{1}{2}x)y'' - (3x+1)y' + (3 + \frac{1}{x})y = 0$ given that one solution is $y = x$.

(Admittedly this problem looks complicated, but I've chosen it so that some of the complications cancel out after a while. Your final answer should be fairly simple, if you do it correctly. So *please* do the algebra carefully.)

Solution. Look for an answer of the form $y = ux$. Then

$$\begin{aligned} y &= ux \\ y' &= u'x + u \\ y'' &= u''x + 2u' \end{aligned}$$

Plug those into the given problem.

$$(x^2 + \frac{1}{2}x)(u''x + 2u') - (3x+1)(u'x + u) + (3 + \frac{1}{x})(ux) = 0.$$

Rearrange terms.

$$(x^3 + \frac{1}{2}x^2)u'' + (2x^2 + x - 3x^2 - x)u' + (-3x - 1 + 3x + 1)u = 0.$$

The u terms cancel out. $\quad (x^3 + \frac{1}{2}x^2)u'' + (2x^2 + x - 3x^2 - x)u' = 0.$

Simplify what's left. $\quad (x^3 + \frac{1}{2}x^2)u'' - x^2u' = 0.$
Getting this far is worth 5 points.

$$(x + \tfrac{1}{2})u'' - u' = 0.$$

Substitute $u' = v$ and $u'' = v' = \dfrac{dv}{dx}$. $\quad (x + \frac{1}{2})\dfrac{dv}{dx} - v = 0.$

Separate variables. $\quad \dfrac{dv}{v} = \dfrac{dx}{x + \frac{1}{2}}.$

Getting this far is worth 6 points.

Integrate both sides. $\quad \ln|v| = \ln|x + \frac{1}{2}| + c_1.$
Getting this far is worth 7 points.

Exponentiate both sides. Change constants to eliminate the absolute values. $\quad v = c_2(x + \frac{1}{2}).$

Getting this far is worth 8 points. $\quad u' = c_2(x + \frac{1}{2})$

Integrate both sides. $\quad u = c_2(\frac{1}{2}x^2 + \frac{1}{2}x) + c_3$
Getting this far is worth 9 points. $\quad u = a(x^2 + x) + b.$

The solution is $y = ux$, so

$$\boxed{y = a(x^3 + x^2) + bx}.$$

Alternate method: Usually I avoid complicated *formulas*, and instead teach the equivalent *procedures*; it is my experience that most students fare better with these. However, some students insist on using formulas. When y_1 is a given solution of the standard form equation $y'' + P(x)y' + Q(x)y = 0$, then the general solution can be found by this formula (see page 141 of your textbook):

$$y_2(x) \quad = \quad y_1(x) \int \frac{\exp\left(-\int P(x)dx\right)}{[y_1(x)]^2} dx.$$

(*Both* constants of integration are needed, and keeping track of the constants is probably the trickiest aspect of using this recipe.) Here is how that may be applied in the present problem. First, $P(x)$ is *not* equal to the coefficient of y' in the given equation, since that equation is not in standard form — i.e., it does not have a coefficient of 1 for y''. To identify $P(x)$, we need to divide the given equation through by its coefficient of y''. That yields

$$y'' + \underbrace{\frac{-3x - 1}{x^2 + \frac{1}{2}x}}_{P(x)} y' + \underbrace{\frac{3 + \frac{1}{x}}{x^2 + \frac{1}{2}x}}_{Q(x)} y = 0.$$

Next compute

$$-\int P(x)dx = \int \frac{3x+1}{x^2+\frac{1}{2}x}dx = \int \left(\frac{A}{x} + \frac{B}{x+\frac{1}{2}}\right) dx$$
$$= \int \left(\frac{2}{x} + \frac{1}{x+\frac{1}{2}}\right) dx = 2\ln|x| + \ln|x+\frac{1}{2}| + c_1.$$

where we have identified the constants $A = 2$ and $B = 1$ by the method of partial fractions (not shown here). Then compute

$$\exp(-\int P(x)dx) = c_2 x^2 (x + \frac{1}{2}) = c_3 x^2 (2x+1).$$

Finally, with $y_1 = x$, we compute

$$\begin{aligned} y_2(x) &= x\int \frac{ax^2(2x+1)}{x^2}dx = x\int a(2x+1)dx \\ &= x\left[a(x^2+x)+b\right] = a(x^3+x^2)+bx \end{aligned}$$

which is the same solution as given earlier. I charged 2 points for loss or misuse of either of the two arbitrary constants.

(8 points) Find the particular solution of $y''' - 6y'' + 9y' = 0$ that satisfies $y(0) = 5$, $y'(0) = 0$, $y''(0) = 9$.

Solution. This is a constant coefficient equation with all roots real, but one repeated. The associated polynomial is $k^3 - 6k^2 + 9k = k(k-3)^2$. Hence the roots are 0, 3, 3. (Not 0, 3. Not 0, -3, -3. Not 0, 3, -3.) Getting that far correctly was worth 2 points.

If you understood this type of problem, you could just write down the form of the answer now, with no further computation:

$$y = a + be^{3x} + cxe^{3x}.$$

Getting that far correctly was worth 6 points.

It remains only to find a, b, c. Now compute

$$\begin{array}{rclcrclcl} y &=& a + be^{3x} + cxe^{3x} & & y(0) &=& a+b & \stackrel{?}{=} & 5 \\ y' &=& (3b+c)e^{3x} + 3cxe^{3x} & & y'(0) &=& 3b+c & \stackrel{?}{=} & 0 \\ y'' &=& (9b+6c)e^{3x} + 9cxe^{3x} & & y''(0) &=& 9b+6c & \stackrel{?}{=} & 9 \end{array}$$

To find a, b, c, we must solve the rightmost colum of equations. The equations for $y'(0)$ and $y''(0)$ together yield $c = 3$ and $b = -1$; then plugging the latter into the equation for $y(0)$ yields $a = 6$. Thus the answer is $\boxed{y = 6 - e^{3x} + 3xe^{3x}}$.

Partial credit: I gave

- 3 points for any answer of the form $y = a + be^{qx} + ce^{rx}$ where q, r were some constants, or 4 points if also $a + b + c = 5$.
- 2 points for any answer of the form $y = ae^{px} + be^{qx} + ce^{rx}$ where p, q, r were some constants, or 3 points if also $a + b + c = 5$.
- 3 points for any answer of the form $y = a + be^{3x}$.

Alternate method: A few students determined correctly, by one method or another, that one solution was $y = e^{3x}$, and then tried to use reduction of order. *If carried out correctly,* that method would have worked, though it's not the easiest way to do this problem. Here are the relevant steps:

$$\begin{array}{rcllll} y & = & ue^{3x} & & & \\ y' & = & u'e^{3x} & +3ue^{3x} & & \\ y'' & = & u''e^{3x} & +6u'e^{3x} & +9ue^{3x} & \\ y''' & = & u'''e^{3x} & +9u''e^{3x} & +27u'e^{3x} & +27ue^{3x} \end{array}$$

Plug those into the given problem. We get

$$(u''' + 9u'' + 27u' + 27u)e^{3x} - 6(u'' + 6u' + 9u)e^{3x} + 9(u' + 3u)e^{3x} = 0.$$

Divide out e^{3x}.

$$(u''' + 9u'' + 27u' + 27u) - 6(u'' + 6u' + 9u) + 9(u' + 3u) = 0.$$

Rearrange terms:

$$u''' + (9 - 6)u'' + (27 - 36 + 9)u' + (27 - 54 + 27)u = 0$$

$$u''' + 3u'' = 0.$$

I gave 5 points for getting this far. Now continue with reduction of order: Substitute $u' = v$, $u'' = v'$, $u''' = v''$. Thus

$$v'' + 3v' = 0.$$

That has associated polynomial $k(k + 3) = 0$, hence roots $k = 0$ and $k = -3$, hence solution $u' = v = a + be^{-3x}$. Integrating yields $u = ax + b_1e^{-3x} + c$. Then $y = ue^{3x}$ yields $y = axe^{3x} + b_1 + ce^{3x}$. Then solve for initial conditions.