Subsequences and Limsups

Some sequences of numbers converge to limits, and some do not. For instance,

\[
1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{5}, \quad \frac{1}{6}, \quad \ldots \quad \text{converges to 0}
\]

\[
3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad 3.14159, \quad \ldots \quad \text{converges to } \pi
\]

\[
1, \quad 3, \quad -\frac{1}{2}, \quad 3.1, \quad \frac{1}{3}, \quad 3.14, \quad \ldots \quad \text{does not converge}
\]

\[
1, \quad -1, \quad 1, \quad -1, \quad 1, \quad -1, \quad \ldots \quad \text{does not converge}
\]

\[
6, \quad 11, \quad -3, \quad 6, \quad 11, \quad -3, \quad \ldots \quad \text{does not converge}
\]

\[
1, \quad -2, \quad 4, \quad -8, \quad 16, \quad -32, \quad \ldots \quad \text{does not converge}
\]

\[
1, \quad 2, \quad 4, \quad 8, \quad 16, \quad 32, \quad \ldots \quad \text{converges to } \infty
\]

provided that we allow \( \infty \) as a possible value for a “limit”. (In some contexts that’s permitted, and in other contexts it’s not.)

A **subsequence** of a given sequence is the sequence we get by skipping some of its terms (but leaving infinitely many unskipped). Even if a sequence is not convergent, it may have a subsequence that is convergent. For instance,

\[
\begin{array}{cccccccc}
1, & 3, & -\frac{1}{2}, & 3.1, & \frac{1}{3}, & 3.14, & \ldots & \text{has convergent subsequences:} \\
1, & -\frac{1}{2}, & \frac{1}{3}, & 3.14, & \ldots & \text{converges to 0;} \\
3, & -\frac{1}{2}, & \frac{1}{3}, & 3.14, & \ldots & \text{converges to } \pi.
\end{array}
\]

\[
\begin{array}{cccccccc}
1, & -1, & 1, & -1, & 1, & -1, & \ldots & \text{has convergent subsequences:} \\
1, & 1, & -1, & 1, & \ldots & \text{converges to 1;} \\
-1, & -1, & -1, & -1, & \ldots & \text{converges to } -1.
\end{array}
\]

\[
\begin{array}{cccccccc}
6, & 11, & -3, & 6, & 11, & -3, & \ldots & \text{has convergent subsequences:} \\
6, & 6, & \ldots & \text{converges to 6;} \\
11, & 11, & \ldots & \text{converges to 11;} \\
-3, & -3, & \ldots & \text{converges to } -3.
\end{array}
\]

\[
\begin{array}{cccccccc}
1, & -2, & 4, & -8, & 16, & -32, & \ldots & \text{has convergent subsequences:} \\
1, & 4, & 16, & \ldots & \text{converges to } +\infty; \\
-2, & -8, & -32, & \ldots & \text{converges to } -\infty.
\end{array}
\]

Other examples:

1
• The sequence \( \left( \sin \frac{n\pi}{2} : n = 1, 2, 3, \ldots \right) \) has subsequences converging to 1, −1, and 0.

• The sequence \( \sin 1, \sin 2, \sin 3, \sin 4, \ldots, \sin n, \ldots \) has subsequences converging to every number in \([-1, 1]\), though that is quite difficult to prove. Indeed, the sequence \((\sin n)\) is rather erratic. (Here “\(\sin 2\)” is understood to mean the sine of 2 radians, not the sine of 2 degrees or the sine of \(2\pi\) radians.)

• The first few terms (i.e., finitely many) do not affect the convergence of a sequence. For instance,

\[
-3, \ 6, \ 6, \ 6, \ 6, \ 6, \ \ldots \quad \text{converges to 6}
\]

\[
-3, \ 6, \ 11, \ 11, \ 11, \ \ldots \quad \text{converges to 11}
\]

where it is understood that, in the first of those sequences, all terms after the first are 6’s; and in the second sequence, all terms after the first two are 11’s.

Here are a few basic facts about convergent subsequences. (We’ll omit the proofs.)

1. If \((x_n)\) is a sequence of real numbers converging to a limit \(L\), then every subsequence of \((x_n)\) also converges to \(L\).

2. If \((x_n)\) is a sequence of real numbers, then there exists at least one subsequence of \((x_n)\) that converges to a limit (if we allow \(\pm \infty\) as possible limits).

3. Suppose \((x_n)\) is a sequence of real numbers, and \(S\) is the set of all the limits of all the convergent subsequences of \((x_n)\). Then \(S\) has a highest member. That number is called the \textit{limsup} of the sequence \((x_n)\); it is written as \(\limsup_{n \to \infty} x_n\).

4. Some sequences have limits, and some do not, but all have limsups. If \((x_n)\) converges to a limit, then its limsup is equal to its limit.

Here are some examples:

\[
1, \ -\frac{1}{2}, \ \frac{1}{3}, \ -\frac{1}{4}, \ \frac{1}{5}, \ -\frac{1}{6}, \ \ldots \quad \text{has limsup equal to 0}
\]

\[
3, \ 3.1, \ 3.14, \ 3.141, \ 3.1415, \ 3.14159, \ \ldots \quad \text{has limsup equal to } \pi
\]

\[
1, \ 3, \ -\frac{1}{2}, \ 3.1, \ \frac{1}{3}, \ 3.14, \ \ldots \quad \text{has limsup equal to } \pi
\]

\[
1, \ -1, \ 1, \ -1, \ 1, \ -1, \ \ldots \quad \text{has limsup equal to } +1
\]

\[
6, \ 11, \ -3, \ 6, \ 11, \ -3, \ \ldots \quad \text{has limsup equal to 11}
\]

\[
1, \ -2, \ 4, \ -8, \ 16, \ -32, \ \ldots \quad \text{has limsup equal to } \infty
\]

\[
1, \ 2, \ 4, \ 8, \ 16, \ 32, \ \ldots \quad \text{has limsup equal to } \infty
\]
Some of the algebraic properties of limsups are not as good as the corresponding properties of limits. For instance, in calculus you learned that

$$\lim_{n \to \infty} (a_n + b_n) = \left( \lim_{n \to \infty} a_n \right) + \left( \lim_{n \to \infty} b_n \right)$$

whenever all the limits exist. The limsups do not satisfy an analogous property; they merely satisfy

$$\limsup_{n \to \infty} (a_n + b_n) \leq \left( \limsup_{n \to \infty} a_n \right) + \left( \limsup_{n \to \infty} b_n \right).$$

To see that equality might not hold, take for instance $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. Then the sequences $(a_n)$ and $(b_n)$ both have limsup equal to 1, but the sequence $(a_n + b_n)$ has limsup equal to 0.

Exercise. Find the limsup of each of these sequences.

1. $x_n = 1 + (-1)^n$
2. $x_n = \frac{1}{1 + 2^n}$
3. $x_n = \begin{cases} n & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$
4. $1.9, 0, -1, 1.99, 0, -2, 1.999, 0, -3, 1.9999, 0, -4, \ldots$

A few useful limits

- If $r$ is a complex number with $|r| < 1$, then $\lim_{n \to \infty} r^n = 0$.

- $\lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e$

- $\lim_{n \to \infty} \sqrt[n]{n} = \lim_{x \to \infty} x^{1/x} = \lim_{x \to 0^+} x^x = 1$

  (The limits given above can be found in many calculus books, but the following limit is more advanced.)

- Stirling’s formula. $\lim_{n \to \infty} \frac{\sqrt{2\pi n} \cdot n^n}{n! \cdot e^n} = 1$, or $\lim_{n \to \infty} \sqrt{n} \left( \frac{n}{e} \right)^n = \frac{1}{\sqrt{2\pi}}$. 

3
Series of Numbers

The sum of a series is defined to be the limit of its partial sums. For instance, consider the series

\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \]

which can also be written as \( \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n \). Its partial sums are

\[
\begin{align*}
  s_0 &= 1 &= 1 \\
  s_1 &= 1 + \frac{1}{2} &= 1.5 \\
  s_2 &= 1 + \frac{1}{2} + \frac{1}{4} &= 1.75 \\
  s_3 &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} &= 1.875 \\
  s_4 &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} &= 1.9375 \\
  &\vdots & &\vdots \\
\end{align*}
\]

The numbers 1, 1.5, 1.75, 1.875, 1.9375, etc., get closer and closer to 2. We say that “the limit of the series is 2,” or that “the sum of the series is 2,” or that “the series converges to 2.”

Some series converge, as in the example above. Other series do not converge. For instance, consider the series

\[ 1 - 1 + 1 - 1 + 1 - \cdots \]

which can also be written as \( \sum_{n=0}^{\infty} (-1)^n \). Its partial sums are

\[
\begin{align*}
  s_0 &= 1 &= 1 \\
  s_1 &= 1 - 1 &= 0 \\
  s_2 &= 1 - 1 + 1 &= 1 \\
  s_3 &= 1 - 1 + 1 - 1 &= 0 \\
  s_4 &= 1 - 1 + 1 - 1 + 1 &= 1 \\
  &\vdots & &\vdots \\
\end{align*}
\]

The numbers 1, 0, 1, 0, 1, etc., do not get closer and closer to a single number. Thus, the series \( \sum_{n=0}^{\infty} (-1)^n \) does not converge. We say that this series is divergent.

What about the series \( 1 + 1 + 1 + 1 + \cdots ? \) Here the terminology of mathematics is ambiguous, unfortunately. In some contexts we would say that this series “diverges”; in other contexts we would say that this series “converges to \( \infty \).”

Here are a few helpful facts about series, for which we’ll omit the proofs. Suppose that \( b_0, b_1, b_2, b_3, \ldots \) are complex numbers. We consider the series \( \sum_{n=0}^{\infty} b_n = b_0 + b_1 + b_2 + b_3 + \cdots \). Then:
If the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are two series convergent to finite sums and $c$ is a number, then the series $\sum_{n=0}^{\infty} c a_n$ and $\sum_{n=0}^{\infty} (a_n + b_n)$ are both convergent to finite sums, with sums

$$\sum_{n=0}^{\infty} c a_n = c \sum_{n=0}^{\infty} a_n, \quad \sum_{n=0}^{\infty} (a_n + b_n) = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n.$$  

If $\sum_{n=0}^{\infty} b_n$ is convergent to a finite sum, then $\lim_{n \to \infty} b_n = 0$. (Call this the “converges to 0 test.”)

*Sketch of proof.* Let $s_k = b_0 + b_1 + b_2 + \cdots + b_k$. By assumption, $L = \lim_{k \to \infty} s_k$ exists. Then $L = \lim_{k \to \infty} s_{k-1}$ also. But $s_k - s_{k-1} = b_k$. Hence $0 = L - L = \lim_{k \to \infty} b_k$.

On the other hand, if $\lim_{n \to \infty} b_n = 0$, it does not follow that $\sum_{n=0}^{\infty} b_n$ is convergent to a finite sum. Two simple examples are the *harmonic series* $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ and the *telescoping series* $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$.

*(Optional. Here are sketches of proofs of the preceding assertions. To show that the harmonic series has infinite sum, note that $\frac{1}{n} = \int_n^{n+1} \frac{1}{x} \, dx > \int_n^{n+1} \frac{1}{2} \, dx$, hence $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \int_1^{n+1} \frac{1}{2} \, dx = \ln(n+1)$. To show that the indicated telescoping series has $b_n \to 0$, compute

$$\sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{(\sqrt{n+1})^2 - (\sqrt{n})^2}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}}.$$  

If $b_0, b_1, b_2, b_3, \ldots$ is a sequence of real numbers with $b_n \geq 0$ for all $n$, then $\sum b_n$ either converges to a finite limit or converges to $\infty$. We write those two alternatives as $\sum b_n < \infty$ or $\sum b_n = \infty$.

If $0 \leq a_n \leq b_n$ and $\sum b_n < \infty$, then $\sum a_n < \infty$.

The *integral test* is sometimes useful in proving that a series converges. The integral test says that if $0 \leq b_n \leq f_n^{n+1} f(t) \, dt$, and $f$ is some function for which we know that $\int_1^{\infty} f(t) \, dt < \infty$, then $\sum b_n < \infty$. For instance, this test tells us that

$$1 + \frac{1}{2^r} + \frac{1}{3^r} + \frac{1}{4^r} + \cdots < \infty \quad \text{for any constant } r > 1.$$  

Actually, the sum for $r = 2$ is rather famous (among mathematicians):

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$  

but that’s hard to prove.

The *alternating series test* tells us that if $b_1 \geq b_2 \geq b_3 \geq b_4 \geq \cdots \geq 0$ and $\lim_{n \to \infty} b_n = 0$ then

$$b_1 - b_2 + b_3 - b_4 + \cdots \text{ converges to a finite limit.}$$
For instance, this test tells us that
\[ 1 - \frac{1}{2^r} + \frac{1}{3^r} - \frac{1}{4^r} + \frac{1}{5^r} - \cdots \]
converges to a finite limit, for any constant \( r > 0 \).

Again, we actually know the value when \( r = 2 \), though it’s hard to prove:
\[ 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots = \frac{\pi^2}{12}. \]

- If \( (c_n) \) is a sequence of complex numbers and \( \sum |c_n| \) is finite, then \( \sum c_n \) converges to some finite limit. In this case we say that \( \sum c_n \) is **absolutely convergent**. One of the properties of an absolutely convergent series is that we get the same sum if we change the order of the terms. For instance,
\[
\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{9^2} - \frac{1}{10^2} + \cdots \\
= 1 + \frac{1}{3^2} - \frac{1}{2^2} + \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{11^2} - \frac{1}{6^2} + \frac{1}{13^2} + \cdots \\
\]
(That last row consists of two odd terms, then an even term, then two odd terms, then an even term, etc.)

- Not every convergent series is absolutely convergent. For instance, the harmonic series is divergent, but
\[
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \cdots = \ln 2 = 0.69314718 \cdots. \\
\]
(That’s not easy to prove.) A series that is convergent but not absolutely convergent is called **conditionally convergent**. We won’t do much with conditionally convergent series in this course.

**Optional.** A conditionally convergent series has the peculiar property that we can get other sums by rearranging the terms in a different order. For instance, it turns out that
\[
1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \cdots = \frac{3}{2} \ln 2 = 1.03972077 \cdots. \\
\]
(That’s two odd terms, then an even term, then two odd terms, then an even term, etc.) And
\[
1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} + \frac{1}{19} - \frac{1}{8} + \cdots = \infty. \\
\]
(That’s an odd term, an even term, two odd terms, an even term, 3 odd terms, an even term, 4 odd terms, an even term, etc.) That’s a little hard to understand, at first. It’s because we’re getting the positive terms more often, or at a faster rate, than the negative terms. In fact, a conditionally convergent series can be made to sum to any number between \(-\infty\) and \(+\infty\) (including those extremes), by suitable rearrangement. When we add finitely many numbers, we may think of adding them all at the same time; but when we add up an infinite series, we need to think in terms of adding them from left to right. The best metaphor I have yet found is this one: You can adjust your shower temperature to whatever you like, by turning the “hot” knob and the “cold” knob to a suitable balance. If you get hot water at a faster rate than cold water, the combined effect will be hotter than if you get both kinds of water at the same rate.
Geometric series. If \( r \) is a complex number with \(|r| < 1\), and \( a \) is a complex number, then \( \sum_{n=0}^{\infty} ar^n \) converges to a finite limit. In fact, we have
\[
\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1 - r}.
\]

**Sketch of proof.** Let \( s_k = \sum_{n=0}^{k} ar^n = a + ar + \cdots + ar^k \). Then \( rs_k = ar + ar^2 + \cdots + ar^{k+1} = s_k - a + ar^k \).
That is, \((1 - r)s_k = a(1 - r^k)\). Dividing yields \( s_k = a \frac{1 - r^k}{1 - r} \). Now take limits as \( k \to \infty \).

Power Series (a series of functions)

Now, what about a power series? Consider, for instance, the formula
\[
f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots
\]
which can also be written as \( f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \). This formula is used to define a function \( f \). For each choice of the number \( x \), we get a different series of numbers; the sum of that series is used as the definition of \( f(x) \) for that value of \( x \). For instance,
- For \( x = 1 \), \( f(1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2 \)
- For \( x = 0 \), \( f(0) = 0 - 0 + 0 - 0 + 0 - \cdots = 0 \)
- For \( x = \frac{1}{2} \), \( f\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \frac{1}{160} - \cdots = \ln \frac{3}{2} \)
- For \( x = -1 \), \( f(-1) = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \cdots = -\infty \) (or undefined)

(It is not supposed to be evident how I got the numbers \( \ln 2 \) or \( \ln \frac{3}{2} \); just take my word for it that those are the correct values.) It is convenient to leave \( f(x) \) undefined whenever the series does not converge to a finite number; that is, we do not allow \( \pm \infty \) for values of the sum. It turns out that this series has the following sums:
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = \begin{cases} 
\ln(1 + x) & \text{if } |x| < 1 \\
\text{varies} & \text{if } |x| = 1 \\
\text{undefined} & \text{if } |x| > 1
\end{cases}
\]
I’ve only indicated the results for real numbers \( x \), but the formula turns out to be correct for any complex number \( x \) (though we need to reconsider what “ln” means when \( x \) is not a real number). The word “varies” indicates that we get different results for different values of \( x \) in the set \( \{ x : |x| = 1 \} \); the behavior is too complicated for us to try to describe it with a simple rule.
Here is another example; we can prove this one using the techniques discussed earlier.

\[
\sum_{n=0}^{\infty} x^n = \begin{cases} 
\frac{1}{1-x} & \text{if } |x| < 1 \\
\text{undefined} & \text{if } |x| \geq 1
\end{cases}
\]

Again, this is for all complex numbers \(x\), not just for all real numbers.

Here is one more example.

\[
\sum_{n=0}^{\infty} \frac{x^n}{3^n+1} = \frac{1}{2} + \frac{x}{4} + \frac{x^2}{10} + \frac{x^3}{28} + \cdots = \begin{cases} 
\text{convergent} & \text{if } |x| < 3, \\
\text{undefined} & \text{if } |x| \geq 3.
\end{cases}
\]

I don’t actually know a formula for the sum of this series when \(|x| < 3\), but I do know that the series converges to some finite limit for each \(x\) with \(|x| < 3\). Moreover, for any particular number \(x\), I can get a very accurate approximation to the value of \(\sum_{n=0}^{\infty} \frac{x^n}{(3^n+1)}\), by using a computer to calculate the sum of (for instance) the first 100 terms of the series.

Note that the graph of the equation \(|x| = 3\) is a circle in the complex plane; we call it the circle of convergence for this series. In this example, the series converges everywhere inside the circle, and diverges everywhere on or outside the circle. That circle has radius 3; we say that this series has 3 for its radius of convergence.

It can be shown that the series

\[
\sum_{n=0}^{\infty} n!x^n = 1 + x + 2x^2 + 6x^3 + 24x^4 + 120x^5 + \cdots
\]

is divergent for every complex number \(x\) except \(x = 0\). This is an example in which the radius of convergence is 0.

It can be shown that each of the series

\[
\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \cdots
\]

\[
\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots
\]

\[
\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots
\]

is convergent for every complex number \(x\). (Thus, we say that those series have radius of convergence equal to infinity.) Those series, which may be unfamiliar to you, turn out to have sums that are surprisingly (if you haven’t studied power series before) familiar. When \(x\) is a real number, it turns out that those three series converge to the familiar old real-valued functions \(e^x\), \(\cos x\), and \(\sin x\), respectively. When \(x\) is some other complex number, such as \(3i\) or \(7 - 5i\) (i.e., not a real number), then those series do still converge to some complex number, which we find convenient to call \(e^x\), \(\cos x\), or \(\sin x\), respectively, because all the familiar equations about those three functions still work. For instance,

\[(\cos x)^2 + (\sin x)^2 = 1, \quad e^{ix} = \cos x + i\sin x, \quad e^{u+v} = e^u e^v,\]
cos(u + v) = cos u \cos v - \sin u \sin v, \quad \frac{d}{dx} e^x = e^x.

For this reason, we write
\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \]

and we take this as a definition of the functions \( e^x, \cos x, \sin x \). More precisely, it is an extension of the familiar old definitions – these “new” functions agree with the old ones when \( x \) is a real number, but these new formulas also give us values for \( e^x, \cos x, \sin x \) when \( x \) is a complex number that is not real.

The function \( e^x \) can also be written as exp(\( x \)).

Another example:
\[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \begin{cases} \arctan(x) & \text{if } |x| < 1 \\ \text{results vary} & \text{if } |x| = 1 \\ \text{undefined} & \text{if } |x| > 1 \end{cases} \]

The radius of convergence is 1. Again, the function \( \arctan(x) \) takes its familiar old values when \( x \) is a real number. When \( x \) is any other complex number, this gives us a broader definition of \( \arctan \). Actually, the function \( \arctan \) is defined for numbers with \( |x| > 1 \), but the series given above doesn’t converge for those values of \( x \).

The phrase “results vary” is my way of glossing over some complications — for instance, the series for \( \ln(1 + x) \) is convergent when \( x = 1 \) and divergent when \( x = -1 \). The series for \( \arctan(x) \) is convergent when \( x = \pm 1 \), but it is divergent when \( x = \pm i \). But those complications occur on the circle of convergence. For the purposes of this course, we usually won’t need to know what happens on the circle. The most important results happen inside the circle of convergence, and that’s much easier to deal with.

**Theorem.** Suppose that \( b_0, b_1, b_2, b_3, \ldots \) are complex numbers. Define a “number” \( R \) with \( 0 \leq R \leq +\infty \) by this formula:
\[ R = \frac{1}{\limsup_{n \to \infty} \sqrt[n]{|b_n|}} \]

with the understanding that \( R = 0 \) if \( \limsup_{n \to \infty} \sqrt[n]{|b_n|} = +\infty \), and \( R = +\infty \) if \( \limsup_{n \to \infty} \sqrt[n]{|b_n|} = 0 \).

Then the power series
\[ \sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots \]

has radius of convergence equal to \( R \). That is,

- the series converges whenever we plug in for \( x \) some complex number that satisfies \( |x| < R \);
• the series diverges whenever we plug in for \( x \) some complex number that satisfies \( |x| > R \); and
• we’re not making any assertion about what happens when \( |x| = R \). (That case is more complicated and would require a much more complicated theorem.)

That’s not actually hard to prove. If we define \( R \) as above, then for \( |x| < R \) we have \( |b_n x^n| \leq |a_n| \) less than or equal to terms of a geometric series with radius less than 1; and for \( |x| > R \) we have \( |b_n x^n| \not\to 0 \).

**Exercises.** Find the radius of convergence of each of these series:

\[
\begin{align*}
(5) \quad & \sum_{n=0}^{\infty} \frac{z^n}{(-3)^n} \\
(6) \quad & \sum_{n=0}^{\infty} \frac{n z^n}{(n+1)^2} \\
(7) \quad & \sum_{n=0}^{\infty} \frac{n^2 z^n}{2^n} \\
(8) \quad & \sum_{n=0}^{\infty} \frac{2^n z^n}{n!} \\
(9) \quad & \sum_{n=0}^{\infty} \left( n + \frac{1}{n+1} \right) z^n \\
(10) \quad & 1 + \frac{1}{4} z^2 + \frac{1}{16} z^4 + \frac{1}{64} z^6 + \cdots + \frac{1}{2^{2k}} z^{2k} + \cdots
\end{align*}
\]

A function is said to be **analytic** (or more precisely, **analytic at 0**) if it has a power series representation with positive radius.

Here are a few more helpful facts about power series; we’ll omit the proofs:

• If the numbers \( b_n \) are nonzero, and \( \lim_{n \to \infty} \frac{|b_n/b_{n+1}|}{|a_n/a_{n+1}|} \) exists, then that limit is equal to \( R \). (In many cases, that limit is easier to compute than the limsup formula given above. However, the limsup formula always works, whereas in some problems the sequence \( |b_n/b_{n+1}| \) is undefined or doesn’t converge.)

  **Caution:** Do not confuse this with \( \lim_{n \to \infty} \frac{|b_{n+1}/b_n|}{|a_{n+1}/a_n|} \), which comes up in some other discussions and other approaches to this subject. In particular, that expression is used for some other purposes in the textbook’s discussion of this subject. But it gives you the reciprocal of the radius of convergence.
Suppose that \( \sum_{n=0}^{\infty} a_n x^n \) and \( \sum_{n=0}^{\infty} b_n x^n \) are two power series with radii of convergence \( R_1 \) and \( R_2 \), respectively. Then
\[
\sum_{n=0}^{\infty} (a_n + b_n) x^n = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n
\]
at least for all \( x \) with \( |x| < \min\{R_1, R_2\} \). Thus, the sum of two analytic functions is analytic.

Suppose that \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) and \( g(x) = \sum_{n=0}^{\infty} b_n x^n \) are two power series with radii of convergence \( R_1 \) and \( R_2 \), respectively. Then we can multiply, term by term:
\[
f(x)g(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0)x^3 + \cdots
\]
and this series has radius of convergence at least as large as \( \min\{R_1, R_2\} \). Thus, the product of two analytic functions is analytic. That’s rather messy, so we won’t do it this semester, but you should make a mental note of the fact that it can be done. (Someday maybe you’ll need to do it.)

If the series \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) has radius of convergence greater than 0, then the function \( f(x) \) is differentiable everywhere inside the disk of convergence, and its derivative has a power series representation with the same radius of convergence. (Thus, the derivative of an analytic function is analytic.) That representation can be obtained by differentiating term by term:
\[
f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k.
\]
That fact is one of the reasons why power series are convenient in solving some kinds of differential equations.

Any function \( f \) has at most one power series representation \( \sum_{n=0}^{\infty} a_n x^n \) with positive radius of convergence. In fact, we even have a formula for that representation:
\[
a_0 = f(0), \quad a_1 = f'(0), \quad a_2 = \frac{f''(0)}{2}, \quad \cdots, \quad a_n = \frac{f^{(n)}(0)}{n!}, \quad \cdots
\]

**Exercises.** Find the numbers \( c_0, c_1, c_2, c_3 \) in each of these series. (You don’t need to find \( c_4, c_5, \ldots \))

11. \( \cos(2z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \)
12. \( \sin(2z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \)
13. \( \frac{1}{z+2} = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \)
14. \( z^3 + 3z + 7 = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \)