A detailed limit proof

We shall present, in detail, a solution to page 117 problem 37: To prove that

\[
\lim_{x \to 1/10} \frac{1}{x} = 10.
\]

We’ll begin with the discovery process. We are given that \( x \) is near \( \frac{1}{10} \) — i.e., that

\[
\left| x - \frac{1}{10} \right| < \delta,
\]

where \( \delta \) is some small number that we get to specify. We’re trying to prove that \( \frac{1}{x} \) is near 10 — i.e., that

\[
\left| \frac{1}{x} - 10 \right| < \varepsilon,
\]

where \( \varepsilon \) is some small number given to us by someone else. I’ve put a question-mark over the less-than sign, to remind myself that that inequality is one we haven’t established yet. That’s something that we want to force to be true, but we haven’t done so yet. That inequality is our goal.

Now, let’s see how that left side can be rearranged. With a bit of algebra, we have

\[
\left| \frac{1}{x} - 10 \right| = \left| 10 - \frac{1}{x} \right| = 10 \cdot \frac{1}{|x|} \cdot \left| x - \frac{1}{10} \right|
\]

I’ve factored things that way because that last factor is the thing that we already know is small.

(It should be noted that this factoring technique will work on many of this semester’s limit proofs, but not all of them. For instance, this factoring technique would not work in finding a proof of \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \).

So our problem is

\[
\underbrace{10}_{\text{known bounded}} \cdot \left| \frac{1}{x} \right| \cdot \underbrace{\left| x - \frac{1}{10} \right|}_{\text{known small}} < \varepsilon.
\]

We’re trying to get the left side of this inequality to be small. We know that the third factor is small, and the first factor is bounded, so we just need to get the
middle factor bounded — i.e., we need to get
\[ \frac{1}{|x|} < \text{some finite number}. \]

And to accomplish that, evidently we need
\[ x > \text{some positive number}. \]

How can we establish that? All we know is that \( x \) is near \( \frac{1}{10} \). Well, pick some number between 0 and \( \frac{1}{10} \) — for instance, pick \( \frac{1}{20} \). We know that when \( x \) is sufficiently near to \( \frac{1}{10} \), then we’ll have \( x > \frac{1}{20} \). How near would \( x \) have to be to \( \frac{1}{10} \)? An easy computation with inequalities (I’ll omit a step or two here) shows that
\[
\text{if } \left| x - \frac{1}{10} \right| < \frac{1}{20} \text{ then } x > \frac{1}{20}.
\]

Now we’re getting somewhere. Our plan is to impose one or more requirements on \( \delta \), and one of the requirements will be that \( \delta \leq \frac{1}{20} \). Then any \( x \) satisfying \( |x - \frac{1}{10}| < \delta \) will also satisfy \( x > \frac{1}{20} \), hence \( \frac{1}{|x|} < 20 \).

Now let’s take another look at our earlier goal of
\[ 10 \cdot \frac{1}{|x|} \cdot \left| x - \frac{1}{10} \right| < \varepsilon. \]

I’m going to add an intermediate step:
\[ 10 \cdot \frac{1}{|x|} \cdot \left| x - \frac{1}{10} \right| < \left( \text{something else that is easier to work with} \right) \leq \varepsilon. \]

Since we already know \( \frac{1}{|x|} < 20 \) (see previous paragraph), we already know the inequality that has a checkmark over it, in the following line:
\[ 10 \cdot \frac{1}{|x|} \cdot \left| x - \frac{1}{10} \right| \sqrt{20} \cdot \frac{1}{10} \leq \varepsilon. \]

So all we still need to establish is that last inequality with the questionmark above it. And we can rewrite that one as
\[ \left| x - \frac{1}{10} \right| \leq \frac{\varepsilon}{200}, \]

which will be satisfied provided that \( \delta \leq \frac{\varepsilon}{200} \). So that’s our other requirement on \( \delta \), and now we’ve completed the discovery process. Now we just have to rearrange the steps to get a \textit{proof}. Here’s how that goes:
Theorem. \( \lim_{x \to 1/10} \frac{1}{x} = 10. \)

Proof. Let any number \( \varepsilon > 0 \) be given. Then we shall choose

\[ \delta = \min \left\{ \frac{1}{20}, \frac{\varepsilon}{200} \right\}. \]

Now suppose that any number \( x \) is given, that satisfies \( 0 < |x - \frac{1}{10}| < \delta \); it suffices for us to prove that \( |\frac{1}{x} - 10| < \varepsilon \).

Since \( |x - \frac{1}{10}| < \delta \leq \frac{1}{20} \), we have

\[ -\frac{1}{20} < x - \frac{1}{10} < \frac{1}{20} \]
\[ -\frac{1}{20} + \frac{1}{10} < x < \frac{1}{20} + \frac{1}{10} \]
\[ \frac{1}{20} < x < \frac{3}{20} \]

and from \( \frac{1}{20} < x \) we obtain

\[ 0 < \frac{1}{x} < 20. \]

Then

\[ \left| \frac{1}{x} - 10 \right| = 10 \cdot \frac{1}{|x|} \cdot \left| x - \frac{1}{10} \right| < 10 \cdot 20 \cdot \delta \leq 10 \cdot 20 \cdot \frac{\varepsilon}{200} = \varepsilon. \]

Q.E.D.

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A two-person dialogue

PROVER: Today we’re working with \( y = 1/x \). And I assert that when \( x \) gets sufficiently near \( 1/10 \), then \( y \) gets near 10.

CHALLENGER: I don’t believe it. Prove it.

Okay, how close to 10 do you want \( y \) to be?
I want $y$ to be within 0.1 of 10. And I want $x$ to be 1/20; that’s near 1/10. Look, when $x = 1/20$, then $y = 20$, which is far from 10. See, I’ve busted you already!

No, that’s not SUFFICIENTLY near.

What?

I said when $x$ gets SUFFICIENTLY near 1/10. I didn’t say how near is “sufficiently” near.

Okay, so how near is “sufficiently” near?

That depends on how close you want the $y$. You get to choose how close $y$ has to be, and then I tell you how close $x$ has to be.

But you can’t say $x$ has to EQUAL 1/10. That wouldn’t be fair.

You’re right about that. I have to give you a POSITIVE number for the $x$ distance.

Okay, I want $y$ to be within 0.1 of 10.

Okay, in that case I require that the distance from $x$ to 1/10 be less than 1/2000.

That’s pretty close.

Well, I can make it as close as I want, as long as it’s a positive distance. That’s the rule.

Okay, so now what? I’m still not convinced of anything.

Okay, now pick an $x$

I choose $x = 1/20$. 
No, you have to pick an $x$ within $1/2000$ of $1/10$.

Okay, I choose $x = 0.1001$.

Well, in that case I get $y$ approximately equal to 9.99. I win.

Wait, what about if I make $x$ equal to $1/10$ plus $1/2000$?

No, it has to be closer than that.

Okay, make $x$ equal to $1/10$ plus $1/2001$.

In that case I get $y$ approximately equal to 9.95. I still win.

No fair. Okay, what if I want $y$ to be within 0.001 of 10?

In that case I require $x$ to be within 0.000005 of $1/10$.

Hmm... We could do this all day. Is there some way to cover all the cases at once?

Yeah, I’ve written out an algebraic proof. Here, take a look. (Go back a couple of pages.)