Boundary Value Problems — Supplement 1
The Method of Characteristic Curves

This material is taken, in part, from pages 5–6 of our textbook.

Problem: Find $u(x,y)$ satisfying the partial differential equation

$$\frac{\partial u}{\partial x} + p(x,y)\frac{\partial u}{\partial y} = 0 \tag{3}$$

where p is some given function.

The strategy is this: We first solve the related *ordinary* differential equation

$$\frac{dy}{dx} = p(x,y) \tag{4}$$

if we can. (We are only to find exact solutions to *some* first-order ordinary differential equations. However, if we also allow numerical approximations, then we can solve *any* first order ODE.)

Since no initial condition is specified for (4), it has many solutions, which we may view as curves in the xy-plane — i.e., solution curves. Let us write those curves as $\phi(x,y) = C$; each choice of the constant C gives us a different one of the solution curves. These curves are called the **characteristic curves** of the original PDE (3). We now claim that

if $f(s)$ is any continuously differentiable function of one variable, then
$$u(x,y) = f(\phi(x,y))$$
is a solution of problem (3), and moreover u is constant along each characteristic curve.

Since there are many possible choices for f, this gives us many solutions of (3). Of course, our problem is also restricted by some initial or boundary condition, that will narrow down our collection of solutions, perhaps just to one solution.

For purposes of this course, the recipe given above is sufficient. The *proof* of our claim is more technical and is optional, so I'll present it in smaller print here.

Observe that $\phi(x,y) = C$ is the solution of the exact differential equation

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = 0$$

which is therefore equivalent to (4) (though (4) itself generally is not exact, and we generally do not find ϕ by the method of exact equations). Consequently, along each solution curve we have $\frac{dy}{dx} = -\frac{\partial \phi}{\partial x}/\frac{\partial \phi}{\partial y}$. Since also $p(x,y) = \frac{dy}{dx}$, it follows that $p(x,y) = -\frac{\partial \phi}{\partial x}/\frac{\partial \phi}{\partial y}$ along the solution curves.

Now let f be any continuously differentiable function, and define u by $u(x,y) = f(\phi(x,y))$. Then the chain rule tells us

$$\frac{\partial u}{\partial x} = f'(\phi(x,y))\frac{\partial \phi}{\partial x}, \qquad \frac{\partial u}{\partial y} = f'(\phi(x,y))\frac{\partial \phi}{\partial y}.$$

Dividing one of those equations by the other, we get

$$\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y} = \frac{\partial \phi}{\partial x}\Big/\frac{\partial \phi}{\partial y} = -p(x,y),$$

so u is a solution of (3).

Example, page 6 problem 3. The given problem is $\dfrac{\partial u}{\partial x}+x^2\dfrac{\partial u}{\partial y}=0$. The related ODE is $\dfrac{dy}{dx}=x^2$. Solve that by separation of variables; $dy=x^2dx$, so $y=\frac{1}{3}x^3+C$. The solution curves are $y-\frac{1}{3}x^3=C$, so we take $\phi(x,y)=y-\frac{1}{3}x^3$. The solution to the original problem then is $\boxed{u(x,y)=f(y-\frac{1}{3}x^3)}$ for any function f. Let's check the answer:

$$\begin{array}{rrcl} \text{If} & u(x,y) & = & f\left(y-\frac{1}{3}x^3\right), \\ \text{then} & \partial u/\partial x & = & f'\left(y-\frac{1}{3}x^3\right)(-x^2) \\ \text{and} & \partial u/\partial y & = & f'\left(y-\frac{1}{3}x^3\right), \end{array}$$

so $\dfrac{\partial u}{\partial x}=-x^2\dfrac{\partial u}{\partial y}$.

Here is a slightly harder problem (not posed in the book): Solve the initial value problem

$$\frac{\partial u}{\partial x}+x^2\frac{\partial u}{\partial y}, \qquad u(0,y)=\sin y.$$

Answer: As before, we have $u(x,y)=f\left(y-\frac{1}{3}x^3\right)$ for some function x. When $x=0$, this equation gives us $\sin y=u(0,y)=f(y)$. So that tells us what f must be — that is, $f(s)=\sin s$ for all numbers s. Therefore $\boxed{u(x,y)=\sin\left(y-\frac{1}{3}x^3\right)}$.

Homework: problems 12 and 14 from page 6. Hint: You may have to rewrite these problems to get them into the form (3).

Additional examples (not in textbook):

Example 1. $\dfrac{\partial u}{\partial x}+\left[(x-y-4)^2+2\right]\dfrac{\partial u}{\partial y}=0$

Solution. The associated ODE is $\dfrac{dy}{dx}=(x-y-4)^2+2$. That can be solved using the affine substitution $u=x-y-4$ (by methods covered in a first course in ODE's, not shown here). The resulting solution is $x+\arctan(x-y-4)=C$. Thus the function we want is $\phi(x,y)=x+\arctan(x-y-4)$, and the solution to the PDE is $\boxed{u(x,y)=f(x+\arctan(x-y-4))}$.

Example 1a. $\dfrac{\partial u}{\partial x}+\left[(x-y-4)^2+2\right]\dfrac{\partial u}{\partial y}=0$ with additional condition $u(0,y)=y^3$.

Solution: As in the preceding example, we arrive at

$$u(x,y)=f(x+\arctan(x-y-4))\,.$$

Plugging in $x = 0$ yields

$$y^3 \stackrel{?}{=} u(0, y) = f(\arctan(-y - 4)).$$

To find f, substitute $-y-4 = \tan\theta$, or $y^3 = (-4-\tan\theta)^3$. That yields $(-4-\tan\theta)^3 = f(\theta)$, so now we know what f is. Plug that formula for f into our formula for u, with $\theta = x+\arctan(x-y-4)$, to obtain

$$\boxed{u(x, y) = [-4 - \tan\left(x + \arctan(x - y - 4)\right)]^3}.$$

Example 2. $\dfrac{\partial u}{\partial x} + \left[\dfrac{2y}{x} - x^2y^2\right]\dfrac{\partial u}{\partial y} = 0.$

Solution. The associated ODE is $\dfrac{dy}{dx} = \dfrac{2y}{x} - x^2y^2$. Rewrite that as

$$\frac{dy}{dx} - 2x^{-1}y = -x^2y^2$$

and you'll see that it is a Bernoulli equation with $n = 2$; thus it yields a linear equation when we substitute $u = y^{-1}$. The solution to the ODE is $x^2y^{-1} - \dfrac{1}{5}x^5 = C$, so the solution to the original PDE is $\boxed{u(x, y) = f\left(x^2y^{-1} - \frac{1}{5}x^5\right)}$.

Example 3. $\dfrac{\partial u}{\partial x} + \dfrac{20x^3y - 3y^4}{5x^4 - 12xy^3}\dfrac{\partial u}{\partial y} = 0$

Solution. The associated ODE is $\dfrac{dy}{dx} = \dfrac{20x^3y - 3y^4}{5x^4 - 12xy^3}$; this is homogeneous. Its solution is $5x^4y - 3xy^4 = C$. Hence the PDE has solution $\boxed{u(x, y) = f(5x^4y - 3xy^4)}$.

Example 3a. $\dfrac{\partial u}{\partial x} + \dfrac{20x^3y - 3y^4}{5x^4 - 12xy^3}\dfrac{\partial u}{\partial y} = 0$ with boundary/initial condition $u(x, x) = \cos x$.

Solution. As in the preceding example, the general solution of the PDE is $u(x, y) = f(5x^4y - 3xy^4)$. Substituting $y = x$ yields $\cos x = u(x, x) = f(2x^5)$. Substitute $z = 2x^5$ and $x = (z/2)^{1/5}$; this yields $\cos\left(\sqrt[5]{z/2}\right) = f(z)$. Now that we know what f is, we can plug that into the general solution of the PDE, with $z = 5x^4y - 3xy^4$. This yields $\boxed{u(x, y) = \cos\sqrt[5]{\frac{5x^4y - 3xy^4}{2}}}$.

Problem 4. $\dfrac{\partial u}{\partial x} + \dfrac{x - 3}{y + 2}\dfrac{\partial u}{\partial y} = 0$

Problem 5. $\dfrac{\partial u}{\partial x} + \dfrac{2x - y - 2}{2x - y + 1}\dfrac{\partial u}{\partial y} = 0$

Problem 5a. $\dfrac{\partial u}{\partial x} + \dfrac{2x - y - 2}{2x - y + 1}\dfrac{\partial u}{\partial y} = 0$ with additional condition $u(x, x) = \cos(x)$.

Problem 6. $\dfrac{\partial u}{\partial x} + \dfrac{-x+5y}{3x+y}\dfrac{\partial u}{\partial y} = 0$

Problem 6a. $\dfrac{\partial u}{\partial x} + \dfrac{-x+5y}{3x+y}\dfrac{\partial u}{\partial y} = 0$ with additional condition $u(0,y) = \ln y$.

Unfortunately, the textbook switches notation between pages 6 and 108, so we must do likewise. To make the results of page 108 more understandable, let us first restate the results of page 6 in new notation. Let $\alpha(x,y) = \dfrac{1}{p(x,y)}$, replace y with t, and replace ϕ with L. The letter ϕ will be used for something else, below. After some algebraic rearrangements, our recipe takes this form:

> Problem: Find $u(x,t)$ satisfying
>
> (3) $$\frac{\partial u}{\partial t} + \alpha(x,t)\frac{\partial u}{\partial x} = 0$$
>
> where α is some given function.

The strategy is this: We first consider the related ordinary differential equation

(4) $$\frac{dx}{dt} = \alpha(x,t)$$

The solution curves of that equation are called the **characteristic curves** of (3). Let us denote those curves by $L(x,t) = C$, with one curve for each choice of C. Then

> if $f(s)$ is any continuously differentiable function of one variable, then
> $$u(x,t) = f(L(x,t))$$
> is a solution of problem (3), and moreover u is constant along each characteristic curve.

The following material is based mostly on page 108 of our textbook. We now consider another type of problem, slightly different from the preceding one.

> Problem: Find $u(x,t)$ satisfying
>
> $$\begin{cases} (3) \qquad \dfrac{\partial u}{\partial t} + A(u)\dfrac{\partial u}{\partial x} = 0, \\ \\ (\mathrm{IC}) \qquad u(x,0) = \phi(x), \end{cases}$$
>
> where A and ϕ are some given functions.

The initial condition was optional in the previous exercises, but it's essential in this modified problem. The big modification is this: instead of a given, known function $\alpha(x,t)$, we have a function $A(u)$ that depends on the unknown u. We can still reason about $A(u)$ as a function of x and t, viewing it as $A(u(x,t))$, so much of our previous reasoning is still valid. But our methods of computation may have to change slightly, since we don't know yet what u is.

Keep in mind that the functions A and ϕ are known. Here is the recipe for the answer:

> If the equation $x = tA(\phi(C)) + C$ can be solved explicitly for C, obtaining an equation of the form $C = L(x,t)$ for some function L, and if $f(s)$ is any continuously differentiable function of one variable, then $u(x,t) = f(L(x,t))$ is a solution of (3). Moreover, when we plug in $\phi(x) = u(x,0) = f(L(x,0))$, we may determine what f is.

Proof (optional). We modify slightly our earlier reasoning about characteristic curves. We still have u constant along each of its characteristic curves. Those are the solutions ordinary differential equation

$$\frac{dx}{dt} = A(u).$$

Along each of those curves, u is constant, so $A(u)$ is constant, so dx/dt is constant, so the "curve" $x = x(t)$ is actually a straight line with slope $A(u)$. Thus, it is the line $x(t) = tA(u) + x(0)$. Say $x(0)$ takes the value C; we get different straight lines for different choices of C.

Let's look at one of those lines. On that line, the value of u is constant. That is, $u = u(x(t),t) = u(x(0),0) = u(C,0) = \phi(C)$. Therefore $A(u) = A(\phi(C))$, and the line is given by $x = tA(\phi(C)) + C$. If we can rewrite that equation in the form $C = L(x,t)$ for some function L, then we can apply the technique of characteristic curves developed earlier.

Example: page 108 problem 13. The given problem is $u_t + \ln(u)u_x = 0$ with initial condition $u(x,0) = e^x$. Here we have $A(u) = \ln(u)$ and $\phi(x) = e^x$. Hence equation $(*)$ is

$$x = t\ln(e^C) + C.$$

Rewrite that as $x = tC + C$, or as $x = (t+1)C$. Thus $x/(t+1) = C$, so we get $L(x,t) = x/(t+1)$. Then the solution to the PDE is $u(x,t) = f(x/(t+1))$. Plug in the initial condition: $e^x \stackrel{?}{=} u(x,0) = f(x)$, so we must have $f(x) = e^x$. Thus the solution is $\boxed{u(x,t) = \exp\left(x/(t+1)\right)}$.

Additional exercises (not in the textbook):

Problem 7. $u_t + (3u-7)u_x = 0$ with $u(x,0) = 5x + 2$.

Problem 8. $u_t + u^{-1}u_x = 0$ with $u(x,0) = \dfrac{1}{x+3}$.

Problem 9. $u_t + u^{-1}u_x = 0$ with $u(x,0) = 4x$, for $x > 0$. *Hints*: Solving for C involves solving a quadratic equation, though you may have to rewrite the problem before that becomes apparent. Also, as with any quadratic equation, you'll find $\pm$ appearing in your computations, but within a few steps you'll find that only one of the two interpretations ($+$ or $-$) yields a meaningful solution.