Math 234 Test 3, Friday 1 December 2005, 6 pages, 30 points, 75 minutes.

There were two perfect scores. The class average was about 25.57 points out of 30, which is 85.24%; that would be a grade of B if I were not putting the grades on a curve.

(7 points) Find the Legendre series for the function x^4. $\boxed{\text{CIRCLE}}$ your answer.

Solution. We must find constants A_0, A_2, A_4 that satisfy $x^4 = A_0P_0 + A_2P_2 + A_4P_4$. That is, the numbers A_0, A_2, A_4 must satisfy

$$x^4 \quad = \quad A_0 \cdot 1 \quad + \quad A_2 \cdot \frac{1}{2}\left(3x^2 - 1\right) \quad + \quad A_4 \cdot \frac{1}{8}\left(35x^4 - 30x^2 + 3\right).$$

That can be rewritten as

$$x^4 + 0x^2 + 0 \;=\; A_0 \cdot (0x^4 + 0x^2 + 1) + A_2 \cdot \frac{1}{2}\left(0x^4 + 3x^2 - 1\right) + A_4 \cdot \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

and thus the linear algebra problem

$$\begin{aligned} 1 &= && && \frac{35}{8}A_4, \\ 0 &= && \frac{3}{2}A_2 && -\frac{30}{8}A_4, \\ 0 &= A_0 && -\frac{1}{2}A_2 && +\frac{3}{8}A_4. \end{aligned}$$

Solve the first equation for $A_4 = \frac{8}{35}$, then solve the second equation for $A_2 = \frac{4}{7}$, and then solve the third equation for $A_0 = \frac{1}{5}$. Thus the answer is $\boxed{x^4 = \frac{1}{5}P_0 + \frac{4}{7}P_2 + \frac{8}{35}P_4}$. Note that, to express your answer, you should **not** multiply out all those things in that box. If you multiply it out, you'll get $x^4 = \frac{1}{5} \cdot 1 + \frac{4}{7} \cdot \frac{1}{2}\left(3x^2 - 1\right) + \frac{8}{35} \cdot \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$, but when you simplify that you just have $x^4 = x^4$. That might be a good way to check your answer, but it's not a good way to *express* your answer. On the other hand, some students arrived at x^4 =something other than x^4, and circled the right side of that equation. This showed not only a computational error, but also a lack of conceptual understanding of what should be going on in the problem.

An alternate method, slightly harder, is to use the formula given on page 312, $f(x) = \sum_{j=0}^{\infty} A_jP_j$, where $A_j = \frac{2j+1}{2}\int_{-1}^{1} f(x)P_j(x)dx$. That yields

$$\begin{aligned} A_0 &= \tfrac{1}{2}\int_{-1}^{1} x^4 dx &&= \tfrac{1}{5}, \\ A_2 &= \tfrac{5}{2}\int_{-1}^{1} \tfrac{1}{2}(3x^6 - x^4)dx &&= \tfrac{5}{2}\cdot\left(\tfrac{3}{7} - \tfrac{1}{5}\right) &&= \tfrac{4}{7}, \\ A_4 &= \tfrac{9}{2}\int_{-1}^{1} \tfrac{1}{8}\left(35x^8 - 30x^6 + 3x^4\right)dx &&= \tfrac{9}{2\cdot 4}\left(\tfrac{35}{9} - \tfrac{30}{7} + \tfrac{3}{5}\right) &&= \tfrac{8}{35} \end{aligned}$$

if carried out correctly; but I've skipped a lot of arithmetic here, and unfortunately some students did not carry it out correctly.

Note that A_4 is the hardest coefficient to compute by the integration method, but the easiest coefficient by the linear algebra method. Perhaps the optimal method evidently would be a combination of the two methods, but that would probably lead to confusion.

(11 points) Write out the series solution $u(x, y)$ of the problem

$$\begin{cases} \nabla^2 u = (\sin 6x)(\sin 5y) & (0 < x < 2\pi, \quad 0 < y < \pi) \\ u(x,0) = \sin 4x, \quad u(x,\pi) = 0 & (0 < x < 2\pi) \\ u(0,y) = 0, \quad u(2\pi, y) = \sin 3y & (0 < y < \pi) \end{cases}$$

Hint: This may look complicated, but I've chosen these functions so that, if you do the problem correctly, the answer simplifies a great deal, enough so that you can check your answer by plugging it back into the given problem. The next page is blank, to give you additional space. Please $\boxed{\text{CIRCLE}}$ your final answer.

Solution: This is a problem of the type on page 170, figure 1. It is the sum of the solutions to two simpler problems: a zero boundary data problem and a Dirichlet problem.

$$\textbf{(zero boundary data)} \begin{cases} \nabla^2 u = (\sin 6x)(\sin 5y) & (0 < x < 2\pi, \quad 0 < y < \pi) \\ u(x,0) = 0, \quad u(x,\pi) = 0 & (0 < x < 2\pi) \\ u(0,y) = 0, \quad u(2\pi, y) = 0 & (0 < y < \pi) \end{cases}$$

The zero boundary data problem is solved on pages 171–172. We can apply those results with $a = 2\pi$, $b = \pi$, and $f(x,y) = (\sin 6x)(\sin 5y)$. We compute

$$\lambda_{mn} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \frac{m^2}{4} + n^2$$

and

$$\begin{aligned} E_{mn} &= \frac{-4}{ab\lambda_{mn}} \int_0^b \int_0^a f(x,y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} dx dy \\ &= \frac{-4}{2\pi^2 \lambda_{mn}} \left[\int_0^{2\pi} (\sin 6x)(\sin\frac{mx}{2}) dx\right] \left[\int_0^{\pi} (\sin 5y)(\sin ny) dy\right]. \end{aligned}$$

Those integrals can be computed by a straightforward computation, or by a shortcut if you know enough about what you're doing. First the straightforward method:

For the integral in x, make the substitution $x = 2\theta$ and use the fact that $\sin 12\theta \sin m\theta$ is an even function, to obtain

$$\int_0^{2\pi} (\sin 6x)(\sin\frac{mx}{2}) dx = \int_{-\pi}^{\pi} (\sin 12\theta)(\sin m\theta) d\theta = \begin{cases} \pi & m = 12, \\ 0 & m \neq 12. \end{cases}$$

The integral in y doesn't need any substitution:

$$\int_0^{\pi} (\sin 5y)(\sin ny) dy = \frac{1}{2}\int_{-\pi}^{\pi} (\sin 5y)(\sin ny) dy = \begin{cases} \pi/2 & n = 5 \\ 0 & n \neq 5 \end{cases}$$

When $m = 12$ and $n = 5$, we obtain $\lambda_{mn} = 36 + 25 = 61$. Thus

$$E_{mn} = \frac{-4}{2\pi^2\lambda_{mn}}[\int][\int] = \begin{cases} \frac{-1}{61} & \text{if } m = 12 \text{ and } n = 5 \\ 0 & \text{otherwise} \end{cases}$$

and so the solution of the zero boundary data problem is $u(x, y) = \frac{-1}{61}(\sin 6x)(\sin 5y)$. We can check that by plugging it into the problem.

If you really understand what's going on, you can use this easier method to compute the E_{mn}'s: They are to be chosen so that equation (3) on page 172 is satisfied. That is,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)E_{mn}\lambda_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = f(x, y)$$

and in this problem that reduces to

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)E_{mn}\lambda_{mn}\sin\frac{mx}{2}\sin ny = (\sin 6x)(\sin 5y).$$

By inspection, the left side must be zero for all terms except the one term where $sin\frac{mx}{2} = \sin 6x$ and $\sin ny = \sin 12y$ — that is, $E_{mn} = 0$ except when $m = 12$ and $n = 5$. And for that one term,

$$(-1)E_{mn}\lambda_{mn}\sin\frac{mx}{2}\sin ny = (\sin 6x)(\sin 5y)$$

becomes

$$(-1)E_{12,5}\lambda_{12,5}\sin 6x \sin 5y = (\sin 6x)(\sin 5y)$$

hence $E_{12,5} = -1/\lambda_{12,5} = -1/61$.

$$\textbf{(Dirichlet problem)}\begin{cases} \nabla^2 u = 0 & (0 < x < 2\pi, \quad 0 < y < \pi) \\ u(x, 0) = \sin 4x, \quad u(x, \pi) = 0 & (0 < x < 2\pi) \\ u(0, y) = 0, \quad u(2\pi, y) = \sin 3y & (0 < y < \pi) \end{cases}$$

The Dirichlet problem is solved on pages 164–168. We can apply those results with

$$a = 2\pi, \quad b = \pi, \quad f_1(x) = \sin 4x, \quad f_2(x) = 0, \quad g_1(y) = 0, \quad g_2(y) = \sin 3y.$$

Compute the coefficients A_n, B_n, C_n, D_n as on page 167. We have $B_n = C_n = 0$ since f_2 and g_1 are both 0. Compute

$$A_n = \frac{2}{a\sinh\frac{n\pi b}{a}}\int_0^a f_1(x)\sin\frac{n\pi}{a}xdx = \frac{2}{2\pi\sinh\frac{n\pi}{2}}\int_0^{2\pi}\sin 4x\sin\frac{nx}{2}dx$$

Substitute $x = 2t$ and then make use of the fact that $\sin 8t \sin nt$ is an even function; then use the orthogonality relations on page 22.

$$A_n = \frac{2\cdot 2}{2\pi\sinh\frac{n\pi}{2}}\int_0^{\pi}\sin 8t\sin ntdt = \frac{1}{\pi\sinh\frac{n\pi}{2}}\int_{-\pi}^{\pi}\sin 8t\sin ntdt = \begin{cases} 0 & \text{if } n \neq 8 \\ \frac{1}{\sinh 4\pi} & \text{if } n = 8 \end{cases}$$

There's an easier way to arrive at that answer, if you understand what's going on. The A_n's are being computed to satisfy the condition that

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi b - n\pi y}{a} \quad \text{agrees with } f_1(x) \text{ at } y = 0.$$

That is to say,

$$\sum_{n=1}^{\infty} A_n \sin\frac{nx}{2} \sinh\frac{n\pi}{2} \quad \text{agrees with } \sin 4x.$$

Hence our choice of A_n.

Similarly,

$$D_n = \frac{2}{b\sinh\frac{n\pi a}{b}} \int_0^b g_2(x) \sin\frac{n\pi}{b} y dy = \frac{2}{\pi \sinh 2n\pi} \int_0^\pi \sin 3y \sin ny\, dy.$$

$$= \frac{1}{\pi \sinh 2n\pi} \int_{-\pi}^{\pi} \sin 3y \sin nydy = \begin{cases} 0 & \text{if } n \neq 3, \\ \frac{1}{\sinh 6\pi} & \text{if } n = 3. \end{cases}$$

The easier approach: we are choosing the D_n's so that

$$\sum_{n=1}^{\infty} D_n \sinh\frac{n\pi x}{b} \sin\frac{n\pi y}{b} \quad \text{agrees with } g_2(x) \text{ at } x = a = 2\pi.$$

That is to say,

$$\sum_{n=1}^{\infty} D_n \sinh 2\pi n \sin ny \quad \text{agrees with } \sin 3y.$$

Hence our choice of A_n.

Using our values for A_n and D_n, we find from page 168 that the solution of the Dirichlet problem is $u(x,y) = \frac{\sin(4x)\sinh(4(\pi-y))}{\sinh 4\pi} + \frac{\sinh(3x)\sin(3y)}{\sinh 6\pi}$. We can check that by plugging it into the Dirichlet problem.

Combining the two parts, we get this answer:

$$\boxed{u(x,y) = \frac{\sin(4x)\sinh(4\pi - 4y)}{\sinh 4\pi} + \frac{\sinh(3x)\sin(3y)}{\sinh 6\pi} - \frac{1}{61}\sin(6x)\sin(5y)}.$$

(6 points) Evaluate $\int_{-1}^{1} x^2 P_4(x)dx = \boxed{}$ and $\int_{-1}^{1} x^4 P_2(x)dx = \boxed{}$.

Solution. The first integral can be done with essentially no work. We know that $x^2 = c_0 P_0 + c_1 P_1 + c_2 P_2$ for some constants c_0, c_1, c_2 (and we don't need to bother finding those constants). We know that $\int_{-1}^{1} P_i P_j = 0$ whenever $i \neq j$. Hence $\boxed{\int_{-1}^{1} x^2 P_4(x)dx = 0}$.

For the second integral, there are a number of methods available, but probably the easiest is simply to use the formula $P_2(x) = \frac{1}{2}(3x^2 - 1)$ from page 303. Then

$$\int_{-1}^{1} x^4 P_2(x)dx = \frac{1}{2}\int_{-1}^{1}(3x^6 - x^4)dx = \int_{0}^{1}(3x^6 - x^4)dx = \frac{3}{7} - \frac{1}{5} = \boxed{8/35}.$$

(6 points) The textbook gives $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x\right]$. Find an analogous formula for $J_{7/2}(x)$. Please $\boxed{\text{CIRCLE}}$ your answer.

Solution. Page 257 gives the formula above for $J_{5/2}$, as well as the formula

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{1}{x}\sin x - \cos x\right].$$

Now apply equation (6) on page 249,

$$J_{p+1}(x) = \frac{2p}{x}J_p(x) - J_{p-1}(x)$$

with $p = 5/2$. Thus

$$J_{7/2}(x) = \frac{5}{x}J_{5/2}(x) - J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\left\{\frac{5}{x}\left(\frac{3}{x^2} - 1\right) - \frac{1}{x}\right\}\sin x - \left\{\frac{5}{x}\cdot\frac{3}{x} - 1\right\}\cos x\right]$$

which simplifies to

$$J_{7/2}(x) = \boxed{\sqrt{\frac{2}{\pi x}}\left[\left(\frac{15}{x^3} - \frac{6}{x}\right)\sin x - \left(\frac{1}{15x^2} - 1\right)\cos x\right]}.$$