Math 234 Test 2, Thursday 27 October 2005, 6 pages, 30 points, 75 minutes.
There were 5 perfect scores.
The class average was about 26.14/30, or about 87.14%, which is a grade of B-plus.

(7 points) Find $u(x,t)$ satisfying $u_{tt} = u_{xx}$ for $0 < x < 1$ and $t > 0$, with boundary conditions $u(0,t) = u(1,t) = 0$ and initial conditions $u(x,0) = \sin(5\pi x)$ and $u_t(x,0) = 0$.

Solution. This is the problem of page 114–115 with $L = 1$, $c = 1$, $f(x) = \sin(5\pi x)$, $g(x) = 0$. The solution is as on page 119. Obviously $b_n^* = 0$, so the solution will satisfy

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\cos(\lambda_n t) \qquad \text{with} \quad \lambda_n = n\pi.$$

We just need to find the b_n's. The most straightforward method is to figure, as on page 115,

$$b_n = 2\int_0^1 \sin(5\pi x)\sin(n\pi x)dx$$

and I gave 5 points for getting that far correctly. You can evaluate that integral directly with some effort — e.g., referring to the table of integrals at the back of the book — or, you can evaluate it by this shortcut if you understand what you're doing: We know that

$$\sin(5\pi x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\cos(0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x).$$

The left side can be viewed as a summation of constants times $\sin(n\pi x)$'s, with most of the constants equal to 0. Either way, we arrive at

$$b_n = \begin{cases} 1 & (n = 5), \\ 0 & (n \neq 5), \end{cases}$$

and finally $\boxed{u(x,t) = \sin(5\pi x)\cos(5\pi t)}$.

(10 points) Find $u(x,y,t)$ satisfying $\nabla^2 u = 0$ for $0 < x < 1$ and $0 < y < 2$, with boundary conditions

$$\begin{array}{lll} u(x,0) = x & u(x,2) = 0 & (0 < x < 1), \\ u(0,y) = 0 & u(1,y) = y & (0 < y < 2). \end{array}$$

Solution. This is just the problem of page 164, with $a = 1$, $b = 2$, $f_1 = x$, $f_2 = 0$,

$g_1 = 0$, $g_2 = y$. The solution is given on pages 167-168. We compute

$$\begin{aligned} A_n &= \frac{2}{a\sinh(n\pi b/a)}\int_0^a f_1(x)\sin\frac{n\pi x}{a}dx = \frac{2}{\sinh(2n\pi)}\int_0^1 x\sin(n\pi x)dx \\ &= \frac{2}{\sinh(2n\pi)}\left[\frac{\sin(n\pi x)}{n^2\pi^2} - \frac{x\cos(n\pi x)}{n\pi}\right]_0^1 = \frac{2(-1)^{n+1}}{n\pi\sinh(2n\pi)} \end{aligned}$$

$$B_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^a f_2(x)\sin\frac{n\pi x}{a}dx = 0$$

$$C_n = \frac{2}{b\sinh(n\pi a/b)}\int_0^b g_1(y)\sin\frac{n\pi y}{b}dy = 0$$

$$\begin{aligned} D_n &= \frac{2}{b\sinh(n\pi a/b)}\int_0^b g_2(y)\sin\frac{n\pi y}{b}dy = \frac{2}{2\sinh(n\pi/2)}\int_0^2 y\sin\left(\frac{n\pi y}{2}\right)dy \\ &= \frac{2}{2\sinh(n\pi/2)}\left[\frac{\sin(n\pi y/2)}{(n\pi/2)^2} - \frac{y\cos(n\pi y/2)}{n\pi/2}\right]_0^2 = \frac{4(-1)^{n+1}}{n\pi\sinh(n\pi/2)} \end{aligned}$$

and therefore

$$\boxed{u(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left\{\frac{\sin(n\pi x)\sinh(2n\pi - yn\pi)}{\sinh(2n\pi)} + \frac{2\sinh(n\pi x/2)\sin(n\pi y/2)}{\sinh(n\pi/2)}\right\}}.$$

(10 points) Find $u(x, y, t)$ satisfying $u_{tt} = 4\nabla^2 u$ in the square $0 < x < 1$, $0 < y < 1$, for all time $t > 0$ with homogeneous boundary conditions ($u = 0$ along the edges of the square, for all time $t > 0$), and with initial conditions

$$u(x, y, 0) = u_t(x, y, 0) = y\sin(2\pi x).$$

Solution. This is the problem of page 155 with $c = 2$, $a = b = 1$, $f = g = y\sin(2\pi x)$. The solution is as on page 158. Compute $\lambda_{mn} = 2\pi\sqrt{m^2 + n^2}$, and

$$B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$

$$= 4\left[\int_0^1 \sin(m\pi x)\sin(2\pi x)dx\right]\left[\int_0^1 y\sin(n\pi y)dy\right].$$

Those two integrals can be evaluated separately.

For the integral in x, we must treat separately the cases of $m = 2$ and $m \neq 2$. When $m \neq 2$, then

$$\int_0^1 \sin(m\pi x)\sin(2\pi x)dx = \left[\frac{\sin((m-2)\pi x)}{2(m-2)\pi} - \frac{\sin((m+2)\pi x)}{2(m+2)\pi}\right]_0^1 = 0$$

since $\sin(k\pi) = 0$ for any integer k. On the other hand, for $m = 2$ we obtain $\int_0^1 \sin^2(2\pi x)dx = \left[\frac{x}{2} - \frac{\sin 4\pi x}{8\pi}\right]_0^1 = \frac{1}{2}$. Thus $\left[\int_0^1 \sin(m\pi x)\sin(2\pi x)dx\right] = \begin{cases} 0 & (m \neq 2), \\ 1/2 & (m = 2). \end{cases}$

For the integral in y,

$$\int_0^1 y\sin(n\pi y)dy = \left[\frac{\sin n\pi y}{n^2\pi^2} - \frac{y\cos n\pi y}{n\pi}\right]_0^1 = \frac{(-1)^{n+1}}{n\pi}.$$

Combining the results of the preceding paragraphs, we obtain

$$B_{mn} = \begin{cases} 0 & (m \neq 2), \\ \dfrac{2(-1)^{n+1}}{n\pi} & (m = 2). \end{cases}$$

Also, in this problem we have $f(x, y) = g(x, y)$, and so

$$B^*_{mn} = \frac{B_{mn}}{\lambda_{mn}} = \begin{cases} 0 & (m \neq 2), \\ \dfrac{(-1)^{n+1}}{n\pi^2\sqrt{4+n^2}} & (m = 2). \end{cases}$$

Finally, $u = \sum_{m,n=1}^{\infty} (B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t)\sin(m\pi x)\sin(n\pi y)$. Putting all the parts together,

$$\boxed{u(x, y, t) = \sin(2\pi x)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left[\frac{2\cos\left(2\pi t\sqrt{4+n^2}\right)}{\pi} + \frac{\sin\left(2\pi t\sqrt{4+n^2}\right)}{\pi^2\sqrt{4+n^2}}\right]\sin(n\pi y)}$$

(3 points) Find $u(x, t)$ satisfying $u_t = u_{xx}$ for $0 < x < 1$ and $t > 0$, with initial condition $u(x, 0) = 1$ and boundary conditions $u(0, t) = u(1, t) = 1$.

Solution: This is a problem of the type on page 140 with $1 = c = L = T_1 = T_2$ and $f(x) = 1$. We can compute, as on page 141:

$$u(x, t) = u_1(x) + u_2(x, t),$$

where

$$u_1(x) = \frac{T_2 - T_1}{L}x + T_1, \qquad u_2(x, t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2 t}\sin\frac{n\pi x}{L},$$

$$\lambda_n = \frac{cn\pi}{L}, \qquad b_n = \frac{2}{L}\int_0^L [f(x) - u_1(x)]\sin\frac{n\pi x}{L}dx.$$

In the example of this problem, we have $u_1(x) = 1$ and

$$b_n = \frac{2}{1}\int_0^1 [1 - 1]\sin\frac{n\pi x}{1}dx = 2\int_0^1 0dx = 0$$

and $\lambda_n = n\pi$, hence

$$u_2(x) = \sum_{n=1}^{\infty} 0 e^{-n^2\pi^2 t} \sin n\pi x = 0$$

and finally $u(x,t) = u_1$ so $\boxed{u(x,t) = 1}$.

But you could also find that solution by inspection — i.e., glance at the problem, guess the answer, and check it by plugging it in — if you've understood any of the discussion we've had about the physical phenomena being modeled by the PDE.