Math 234 Test 1, Tuesday 27 September 2005, 4 pages, 30 points, 75 minutes.

The high score was 29 points out of 30, achieved by two students. The class average is 23.25 points out of 30, or 77.5%, which ordinarily would be a grade of C+. However, I think this test may have been a little too hard, so I'm going to curve the grades upward a bit at the end of the semester. I haven't yet decided how much.

(6 points) Solve $\dfrac{\partial u}{\partial x} + x^2 y \dfrac{\partial u}{\partial y} = 0$.

Solution. First we solve the related ODE $\dfrac{dy}{dx} = x^2 y$. Use separation of variables:

$$\ln|y| = \int y^{-1} dy = \int x^2 dx = \frac{1}{3}x^3 + c_1$$

hence $y = k\exp(x^3/3)$, or $\exp(-x^3/3)y = k$. The solution to the given partial differential equation is then $\boxed{u(x,y) = f(ye^{-x^3/3})}$. A wide variety of variants are possible; here are a few: $\boxed{u(x,y) = f(y^3 e^{-x^3})}$, $\boxed{u(x,y) = f(y^{-3}e^{x^3})}$, $\boxed{u(x,y) = f(\ln|y| - \frac{1}{3}x^3)}$, $\boxed{u(x,y) = f(x^3 - 3\ln|y|)}$.

I'll use that last formulation to check the answer. If we define u by that equation, and abbreviate $\theta = x^3 - 3\ln|y|$, then $\dfrac{\partial\theta}{\partial x} = 3x^2$ and $\dfrac{\partial\theta}{\partial y} = \dfrac{-3}{y}$, so we get $\dfrac{\partial u}{\partial x} = f'(\theta)\cdot 3x^2$ and $\dfrac{\partial u}{\partial y} = f'(\theta)\cdot\dfrac{-3}{y}$, and thus $\dfrac{\partial u}{\partial x} = -x^2 y\dfrac{\partial u}{\partial y}$, which is equivalent to the PDE given int the problem.

Common errors: I decided to not deduct a point *this time* for omitting the absolute value signs in the logarithm, even though that error does make the answer wrong. Also, I decided to let go the error of writing fractions incorrectly: the expression $1/3x^3$ is ambiguous and wrong, but I let it go *this time.* See

http://www.math.vanderbilt.edu/~schectex/commerrs/

for a discussion of the latter error.

I deducted a point for writing $ye^{-x^3/3}$ either as $y - e^{-x^3/3}$ or as $y^{-x^3/3}$, and I will probably deduct more than a point the next time I see errors of those types; those show more substantial misunderstandings of exponents.

(6 points) $\dfrac{\partial u}{\partial t} + (2u+1)\dfrac{\partial u}{\partial x} = 0$, with $u(x,0) = 2x - 1$.

Solution. In this problem $A(u) = 2u + 1$ and $\phi(C) = 2C - 1$, so the characteristic lines $x = t(A(\phi(C))) + C$ can be rewritten as $x = t(2(2C-1)+1) + C$ which simplifies to $(x+t)/(4t+1) = C$. I gave 3 points for getting this far correctly.

The solution of the PDE therefore takes the form $u(x,t) = f\left(\frac{x+t}{4t+1}\right)$. Now plug in $t=0$, to get $2x-1 = u(x,0) = f(x)$. So the answer finally is any of the following:

$$u(x,t) = \boxed{2\frac{x+t}{4t+1} - 1} = \boxed{\frac{2x-2t-1}{4t+1}} = \boxed{\frac{1}{2}\left(\frac{4x-1}{4t+1} - 1\right)}.$$

Note that the last boxed answer yields $2u+1 = \frac{4x-1}{4t+1}$. To check our answers, compute $u_x = \frac{2}{4t+1}$ and $u_t = \frac{-2(4x-1)}{(4t+1)^2}$, hence $u_t + (2u+1)u_x = 0$, as required.

(10 points) Define the function

$$f(x) = \begin{cases} 0 & -\pi < x < 0, \\ x & 0 < x < \pi, \end{cases}$$

and say f is periodic with period 2π. Find its Fourier series. Express your answer **both** in trigonometric form (sines and cosines) **and** in complex exponential form. Circle both answers. The next page is blank, to provide you with some extra space for computation. *Hint*: If you understand this material well, it is easiest to do the complex version first and then convert it to the trigonometric version.

Solution.

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_0^{\pi} xe^{-inx}dx = \frac{1}{2\pi}\left[\frac{e^{-inx}}{(-in)^2}(-inx-1)\right]_{x=0}^{x=\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-inx}}{-n^2}(-inx-1)\right]_{x=0}^{x=\pi} = \frac{1}{2\pi n^2}\left[e^{-inx}(inx+1)\right]_{x=0}^{x=\pi}$$

$$= \frac{1}{2\pi n^2}\left[e^{-in\pi}(in\pi+1) - e^0(0+1)\right] = \frac{1}{2\pi n^2}\left[(-1)^n(in\pi+1)-1\right]$$

when $n \neq 0$. On the other hand, $c_0 = \frac{1}{2\pi}\int_0^{\pi} xdx = \frac{\pi}{4}$. Thus we get

$$\boxed{f(x) = \frac{\pi}{4} + \frac{1}{2\pi}\sum_{n\neq 0}\frac{(-1)^n(in\pi+1)-1}{n^2}e^{inx}}.$$

This could be written in some other ways. For instance,

$$\boxed{f(x) = \frac{\pi}{4} + \frac{1}{2\pi}\sum_{n\neq 0}\left[\frac{(-1)^n i\pi}{n} + \frac{(-1)^n-1}{n^2}\right]e^{inx}}.$$

Note that

$$(-1)^n - 1 = \begin{cases} 0 & \text{when } n \text{ is even,} \\ -2 & \text{when } n \text{ is odd.} \end{cases}$$

So the answer could also be written as

$$\boxed{f(x) = \frac{\pi}{4} + \frac{i}{2}\sum_{n\neq 0}\frac{(-1)^n}{n}e^{inx} - \frac{1}{\pi}\sum_{k\in\mathbb{Z}}\frac{e^{i(2k+1)x}}{(2k+1)^2}}.$$

To convert to trigonometric form, we begin by combining the n and $-n$ terms. Thus we have

$$\begin{aligned}
f(x) &= \frac{\pi}{4} + \frac{1}{2\pi}\sum_{n\neq 0}\left[\frac{(-1)^n i\pi}{n} + \frac{(-1)^n - 1}{n^2}\right]e^{inx} \\
&= \frac{\pi}{4} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\left\{\left[\frac{(-1)^n i\pi}{n} + \frac{(-1)^n - 1}{n^2}\right]e^{inx} + \left[\frac{(-1)^{-n} i\pi}{-n} + \frac{(-1)^{-n} - 1}{(-n)^2}\right]e^{-inx}\right\} \\
&= \frac{\pi}{4} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\left\{\left[\frac{(-1)^n i\pi}{n} + \frac{(-1)^n - 1}{n^2}\right]e^{inx} + \left[\frac{-(-1)^n i\pi}{n} + \frac{(-1)^n - 1}{n^2}\right]e^{-inx}\right\} \\
&= \frac{\pi}{4} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\left\{\frac{(-1)^n - 1}{n^2}(e^{inx} + e^{-inx}) + \frac{(-1)^n i\pi}{n}(e^{inx} - e^{-inx})\right\}.
\end{aligned}$$

Then we make use of the formulas $e^{inx} + e^{-inx} = 2\cos nx$ and $e^{inx} - e^{-inx} = 2i\sin nx$. Thus we have

$$\boxed{f(x) = \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left\{\frac{(-1)^n - 1}{n^2}\cos nx - \frac{(-1)^n\pi}{n}\sin nx\right\}}$$

$$\text{or} \quad \boxed{f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{(-1)^n - 1}{\pi n^2}\cos nx - \frac{(-1)^n}{n}\sin nx\right\}}.$$

Again, if we wish we can make use of the fact that

$$(-1)^n - 1 = \begin{cases} 0 & \text{when } n \text{ is even,} \\ -2 & \text{when } n \text{ is odd.} \end{cases}$$

Thus we obtain

$$\boxed{f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\cos(2k+1)x}{(2k+1)^2} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx}.$$

In my solution, I computed the complex formulas first, and then converted to trigonometric formulas. I suggested that route because I thought it was simplest. However, perhaps it was just as easy (or for some students, easier) to compute the trigonometric version directly. Here are the computations:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{2\pi}\int_0^{\pi} t\,dt = \frac{\pi}{4};$$

and, for $n > 0$,

$$\begin{aligned} a_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \tfrac{1}{\pi}\int_0^{\pi} t\cos nt\,dt = \tfrac{1}{\pi}\left[\tfrac{\cos nt}{n^2} + \tfrac{t\sin nt}{n}\right]_{t=0}^{t=\pi} \\ &= \tfrac{1}{\pi}\left[\tfrac{\cos n\pi - \cos 0}{n^2} + \tfrac{\pi\sin n\pi - 0\sin 0}{n}\right] = \tfrac{(-1)^n - 1}{\pi n^2} \end{aligned}$$

$$\begin{aligned} b_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \tfrac{1}{\pi}\int_0^{\pi} t\sin nt\,dt = \tfrac{1}{\pi}\left[\tfrac{\sin nt}{n^2} - \tfrac{t\cos nt}{n}\right]_{t=0}^{t=\pi} \\ &= \tfrac{1}{\pi}\left[\tfrac{\sin n\pi - \sin 0}{n^2} - \tfrac{\pi\cos n\pi - 0\cos 0}{n}\right] = \tfrac{-(-1)^n}{n} \end{aligned}$$

Note that a_n and b_n are *numbers*, which depend only on n. They are not functions of x or of t. If you ended up with functions of x or t for your coefficients (as one student did), then you not only made a computational error, but showed a substantial conceptual error.

Common errors: The most common errors were sign errors (plus for minus or minus for plus), for which I charged 1 point.

Some students did not know what notation to use, where I have used $\sum\limits_{n\neq 0}$. I generally did not deduct points for poor notation, since this particular notational situation is one that I had not discussed in class. For instance, one student wrote

$$f(x) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} \frac{(-1)^n(in\pi+1)-1}{n^2}e^{inx} \quad \text{for } \neq 0, \qquad c_0 = \frac{\pi}{4}.$$

That notation is *wrong*, but I gave full credit for it anyway, just this time.

On the other hand, some students committed the much more grievous error of treating the $n = 0$ term just the same as any other term. The expression $\dfrac{(-1)^n(in\pi+1)-1}{n^2}$ is nonsense when $n = 0$, and that's something you should notice regardless of how well you understand Fourier series. I'm sure we had at least one comparable example in class with trigonometric series. We probably did not have a comparable example with complex series; for that reason I was somewhat lenient: I charged only 2 points for this error, instead of the 4 point penalty that nonsense really deserves.

Another common error was to omit some needed braces after a summation sign, in an expression such as $\sum_{n=1}^{\infty}\left\{\frac{(-1)^n-1}{\pi n^2}\cos nx - \frac{(-1)^n}{n}\sin nx\right\}$. Grammatically, the symbol Σ acts like $+$ or $-$, so its scope only extends to the next $+$ or $-$ sign. The expression

$$\sum_{n=1}^{\infty}\frac{(-1)^n-1}{\pi n^2}\cos nx - \frac{(-1)^n}{n}\sin nx$$

(written by several students) is *wrong*, but I didn't deduct any points for it this time. Most mathematicians would read that as $\left\{\sum_{n=1}^{\infty}\frac{(-1)^n-1}{\pi n^2}\cos nx\right\} - \left\{\frac{(-1)^n}{n}\sin nx\right\}$, and then would declare it nonsense, since x is a bound variable in the first pair of braces and a free variable in the second pair of braces.

(8 points) Define the function

$$f(x) = \begin{cases} x & 0 < x < \frac{1}{2}\pi, \\ 0 & \frac{1}{2}\pi < x < \pi. \end{cases}$$

Find the sine series for f, agreeing with f on the interval $(0, \pi)$.

Solution. The correct analysis is to first extend f to an odd function defined on $(-\pi, \pi)$, and then to a function defined on all of $\mathbb{R}$ which is periodic with period 2π. Thus the formulas on page 50 are applicable, with $p = \pi$. The sine coefficients are

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi/2} x \sin nx dx = \frac{2}{\pi}\left[\frac{\sin nx}{n^2} - \frac{x\cos nx}{n}\right]_{x=0}^{x=\pi/2}$$

$$= \frac{2}{\pi n^2}\left(\sin\frac{n\pi}{2} - \sin 0\right) - \frac{2}{\pi n}\left(\frac{\pi}{2}\cos\frac{n\pi}{2} - 0\cos 0\right) = \frac{2}{\pi n^2}\sin\frac{n\pi}{2} - \frac{1}{n}\left(\cos\frac{n\pi}{2}\right)$$

$$= \begin{cases} \dfrac{2(-1)^{(n-1)/2}}{\pi n^2} = \dfrac{2(-1)^k}{\pi(2k+1)^2} & \text{when } n = 2k+1 = \text{odd}, \\[2ex] \dfrac{(-1)^{(n/2)+1}}{n} = \dfrac{(-1)^{k+1}}{2k} & \text{when } n = 2k = \text{even}. \end{cases}$$

Hence the sine series can be written in any of the following ways:

$$\boxed{f(x) = \sum_{n=1}^{\infty}\left(\frac{2}{\pi n^2}\sin\frac{n\pi}{2} - \frac{1}{n}\cos\frac{n\pi}{2}\right)\sin nx}$$

$$\boxed{f(x) = \sum_{k=0}^{\infty}\frac{2(-1)^k}{\pi(2k+1)^2}\sin((2k+1)x) + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k}\sin(2kx)}$$

$$\boxed{\begin{aligned} f(x) &= \frac{2}{\pi}\left(\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2} + \frac{\sin 9x}{9^2} - \cdots\right) \\ &\quad + \left(\frac{\sin 2x}{2} - \frac{\sin 4x}{4} + \frac{\sin 6x}{6} - \frac{\sin 8x}{8} + \frac{\sin 10x}{10} - \cdots\right) \end{aligned}}$$

Common errors. For reasons that I haven't figured out yet, a couple of students computed with $p = \pi/2$. If carried out correctly, that would yield a sine series of $f(x) = \sum_{n=1}^{\infty} b_n \sin 2nx$, with

$$b_n = \frac{2}{p}\int_0^p f(x)\sin 2nx\,dx = \frac{4}{\pi}\int_0^{\pi/2} x\sin 2nx = \frac{(-1)^{n+1}}{n},$$

and thus the sine series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin 2nx$. I gave 5 points for that expression, or fewer points for something slightly different (i.e., if some error was made en route to that expression). Note that that series agrees with the given function only on the interval $(0, \pi/2)$, not on $(0, \pi)$. To see what it adds up to on the rest of the interval, follow this procedure: Start with the function $f(x)$ that was given in the problem; restrict it to the interval $(0, \pi/2)$; extend it to $(-\frac{\pi}{2}, \frac{\pi}{2})$ as an odd function; and then extend it to the real line as a function periodic with period π. We obtain this function:

$$g(x) \;=\; \begin{cases} x & (0 < x < \frac{\pi}{2}), \\ x - \pi & (\frac{\pi}{2} < x < \pi). \end{cases}$$