Boundary Value Problems
Homework Set 15
due Thursday Dec 8

Page 407, problems 1, 3. Find the Fourier transform.

Problem 1.

$$f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Solution.

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{0} (-1) e^{-i\omega x} dx + \int_{0}^{1} e^{-i\omega x} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{-e^{-i\omega x}}{-i\omega} \right]_{x=-1}^{x=0} + \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\omega x}}{-i\omega} \right]_{x=0}^{x=1} = \frac{1}{\sqrt{2\pi}} \left\{ \frac{-1 + e^{i\omega}}{-i\omega} + \frac{e^{-i\omega} - 1}{-i\omega} \right\} = \boxed{\frac{e^{i\omega} - 2 + e^{-i\omega}}{-i\omega\sqrt{2\pi}}}$$

$$= \frac{(e^{i\omega/2} - e^{-i\omega/2})^2}{-i\omega\sqrt{2\pi}} = \frac{\left(2i \sin \frac{\omega}{2}\right)^2}{-i\omega\sqrt{2\pi}} = \boxed{\frac{4 \sin^2 \frac{\omega}{2}}{i\omega\sqrt{2\pi}}} = \boxed{\frac{2 - 2\cos\omega}{i\omega\sqrt{2\pi}}} = \boxed{\frac{i(\cos\omega - 1)}{\omega} \sqrt{\frac{2}{\pi}}}$$

Problem 3.

$$f(x) = \begin{cases} \sin x & |x| \le \pi \\ 0 & \text{elsewhere} \end{cases}$$

Solution.

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} (\sin x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \frac{e^{ix} - e^{-ix}}{2i} e^{-i\omega x} dx$$

$$= \frac{1}{2i\sqrt{2\pi}} \int_{-\pi}^{\pi} \left(e^{(i-i\omega)x} - e^{(-i-i\omega)x} \right) dx = \frac{1}{2i\sqrt{2\pi}} \left[\frac{e^{(i-i\omega)x}}{i - i\omega} - \frac{e^{(-i-i\omega)x}}{-i - i\omega} \right]_{x=-\pi}^{x=\pi}$$

$$= \frac{1}{2i\sqrt{2\pi}} \left[\frac{e^{(i-i\omega)\pi} - e^{(i-i\omega)(-\pi)}}{i - i\omega} - \frac{e^{(-i-i\omega)\pi} - e^{(-i-i\omega)(-\pi)}}{-i - i\omega} \right] \qquad \left(\begin{array}{c} \text{now use the fact} \\ \text{that } e^{i\pi} = e^{-i\pi} = -1 \end{array} \right)$$

$$= \frac{1}{2i\sqrt{2\pi}} \left[\frac{-e^{-i\omega\pi} + e^{i\omega\pi}}{i - i\omega} - \frac{-e^{-i\omega\pi} + e^{i\omega\pi}}{-i - i\omega} \right] = \frac{1}{2i\sqrt{2\pi}} \left[e^{i\omega\pi} - e^{-i\omega\pi} \right] \left[\frac{1}{i - i\omega} + \frac{1}{-i - i\omega} \right]$$

$$= \frac{1}{2i\sqrt{2\pi}} [2i \sin \pi\omega] \frac{1}{i} \left[\frac{1}{1 - \omega} + \frac{1}{-1 - \omega} \right] = \frac{1}{2i\sqrt{2\pi}} [2i \sin \pi\omega] \frac{1}{i} \left[\frac{-2\omega}{\omega^2 - 1} \right] = \boxed{\frac{i\omega \sin \pi\omega}{\omega^2 - 1} \sqrt{\frac{2}{\pi}}}$$