The Volume of a Generalized Cone

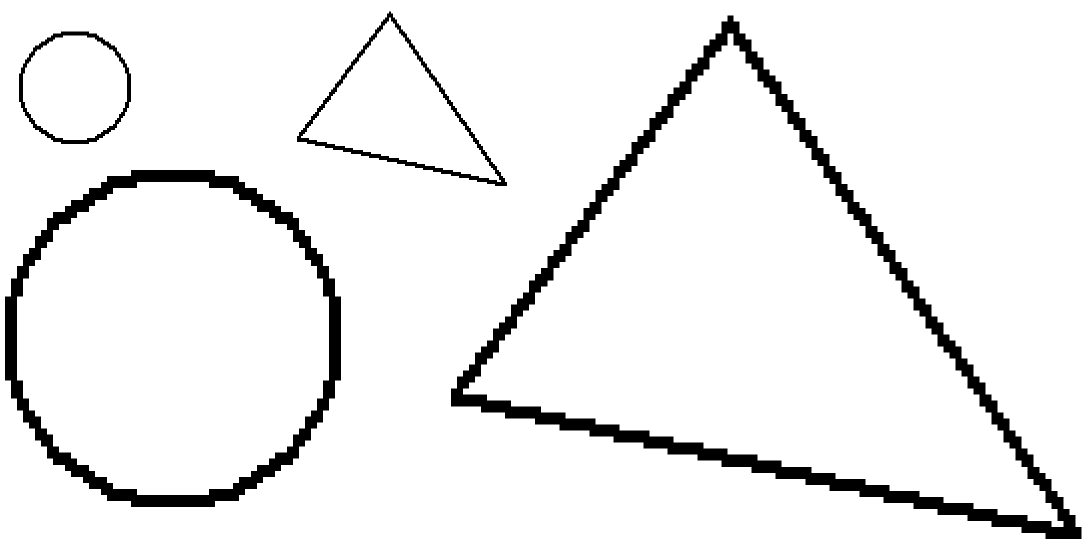
You should already know from geometry that

(1) | the area of a triangle is $\frac{1}{2}$(base length)(height). |

We're going to prove an interesting generalization of that fact to 3 dimensions. But first we need one other fact from geometry:

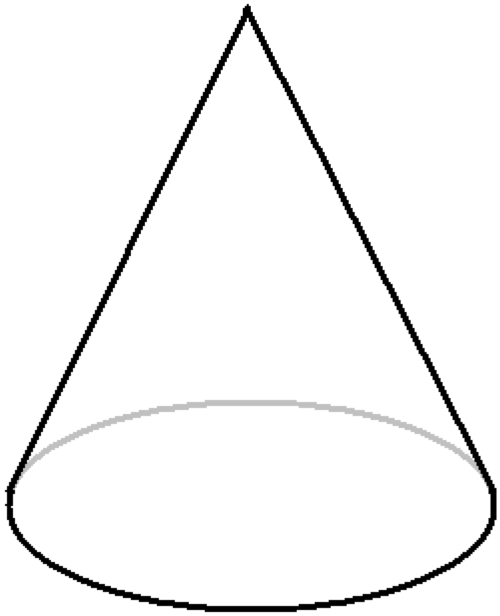
(2) | The ratio of the areas of two similar figures is the square of the ratio of any one-dimensional measurement in the figures. |

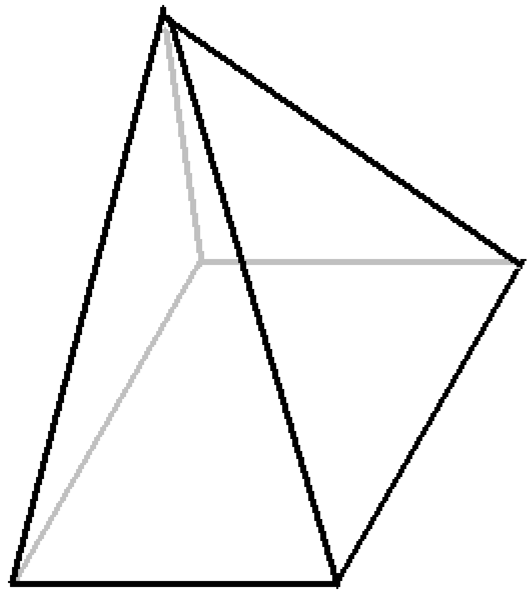
That means, for instance,

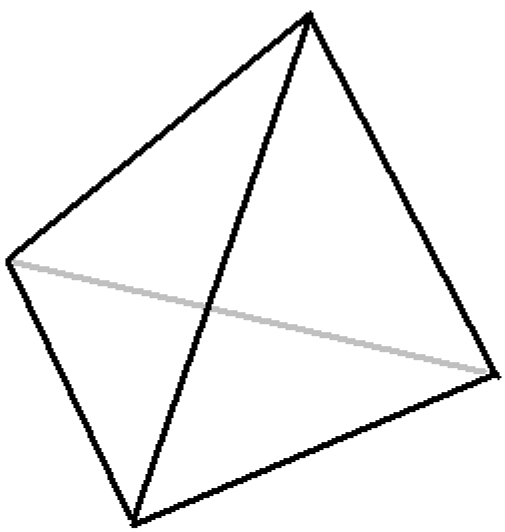


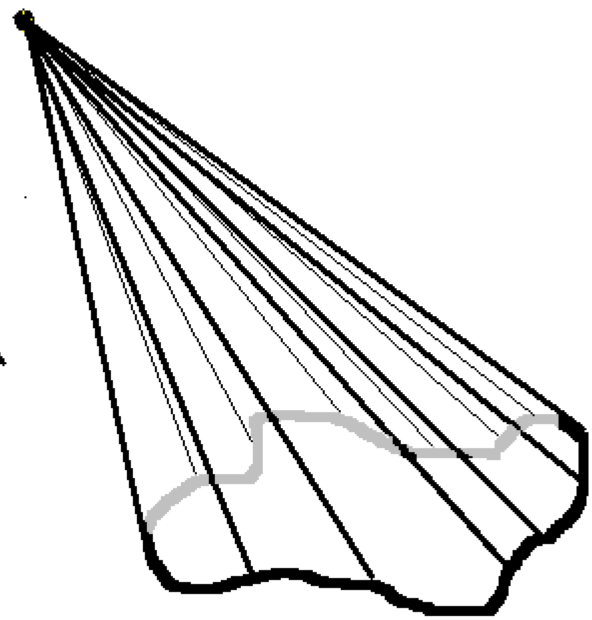
if you compare two circles, and the bigger radius is 3 times the smaller radius, then the bigger area is 9 times the smaller area. Or if the bigger triangle is 3 times as wide as the smaller triangle, then it also is 3 times as tall, and so it is 9 times as big in area.

Now, let's go on to three dimensions. What most people call a "cone" is what a mathematician calls a **right circular cone**. What a mathematician calls a "cone" is something more general, which includes the right circular cone as a special case. It also includes, as other special cases, a pyramid, a tetrahedron, and certain other weird shapes:







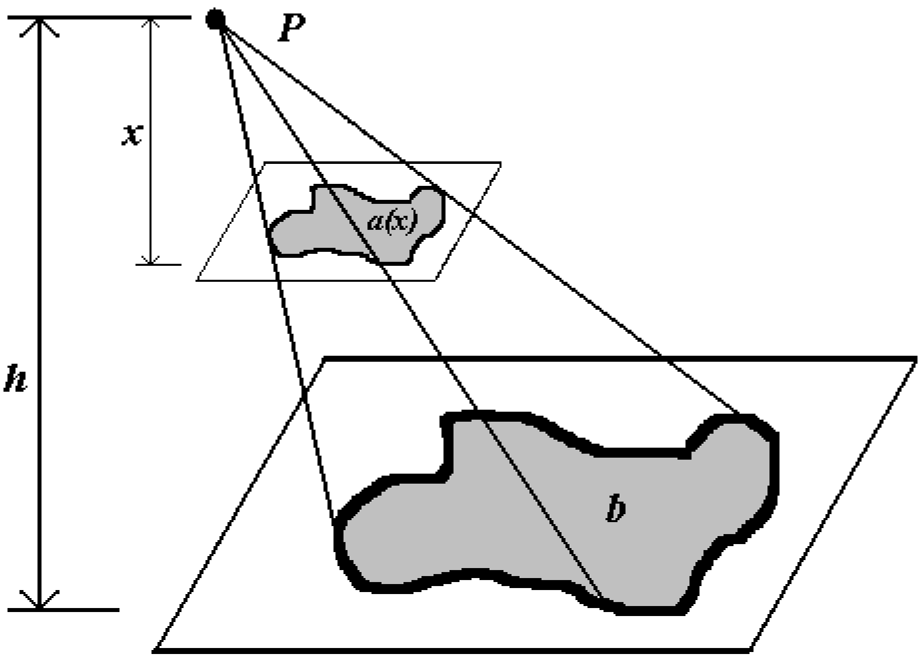


Here's what all these figures have in common: connect all the points in some planar region to some point outside that region; the resulting solid is what mathematicians call a **cone**. And we'll prove

$$\text{(3)} \qquad \boxed{\begin{array}{c}\text{the volume of a cone is} \\ \tfrac{1}{3}\text{(base area)(height).}\end{array}}$$

Note how that is analogous to formula (1). But it includes as special cases the formulas for the volumes of a right circular cone, a pyramid, a tetrahedron, and other odd shapes.

To begin, we must analyze that last picture. Call the apex point P. The base region R (for "region"), with area b (for "base") lies in a certain plane, and the way that we define the height h of the cone is the distance from P to that plane. In other words, drop a perpendicular from P to that plane, and measure its length. The perpendicular might not actually hit the region R — that is, the apex P does not have to be directly over a part of the region R.



We're going to calculate the volume of the cone by the same method we've been calculating all our volumes: Cut it up into thin slices. Let's consider the slice that lies in the plane that has distance x from the apex point P, where x is between 0 and h. Say that slice has area $a(x)$. Then it has volume $a(x)\Delta x$, if we're only counting finitely many slices; or it has volume $a(x)dx$, when we take the limit as the number of slices goes to infinity and the thickness of the slices becomes infinitesimal. Therefore the volume of the entire cone is $\int_0^h a(x)dx$.

But by our earlier observation on proportionality, we can see that

$$\frac{a(x)}{b} = \left(\frac{x}{h}\right)^2,$$

and therefore the volume of the cone is

$$\int_0^h b\frac{x^2}{h^2}\,dx \;=\; \frac{b}{h^2}\int_0^h x^2 dx \;=\; \frac{b}{h^2}\left[\frac{1}{3}x^3\right]_0^h \;=\; \frac{b}{h^2}\cdot\frac{1}{3}h^3 \;=\; \frac{1}{3}bh.$$

That completes the proof of (3).