Name (please print):
Math 150B Quiz 8, Thursday 2 April 2009, 1 page, 15 points, 15 minutes

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(5 points) $\displaystyle\int \frac{6\,dx}{x^2-1} =$

Solution. Factor $x^2-1=(x-1)(x+1)$. Then

$$\frac{6}{x^2-1} = \frac{A}{x-1}+\frac{B}{x+1}$$

I gave 2 points just for getting that far. Then

$$6 = (x+1)A+(x-1)B$$

Then $x=1$ yields $6=2A$, so $A=3$; and $x=-1$ yields $6=-2B$, so $B=-3$. Then

$$\int\left(\frac{3}{x-1}-\frac{3}{x+1}\right)dx = \boxed{3\ln|x-1|-3\ln|x+1|+C} = \boxed{3\ln\left|\frac{x-1}{x+1}\right|+C}$$

I deducted 1 point for omitting the absolute values, or for a $\pm$ error, or for replacing the 3 with something else (e.g., 1/2).

Some students seemed to think that $\displaystyle\int\frac{dx}{f(x)}$ is equal to $\ln|f(x)|+C$, regardless of what $f(x)$ is; but that's wrong, and I gave 0 credit for it. (The correct formula is $\displaystyle\int\frac{f'(x)}{f(x)}dx = \ln|f(x)|+C$.) And some students seemed to think that this problem had something to do with $\displaystyle\int\frac{dx}{x^2+1} = \arctan x + C$; again, zero credit.

Some students tried doing this the hard way: Substitute $x=\sec\theta$ and $x^2-1=\tan^2\theta$ and $dx=\sec\theta\tan\theta d\theta$; then

$$\int\frac{6dx}{x^2-1} = 6\int\frac{\sec\theta\tan\theta\,d\theta}{\tan^2\theta} = 6\int\frac{\sec\theta}{\tan\theta}d\theta = 6\int\csc\theta\,d\theta$$

and I gave 2 points for going that far; no students succeeded in going any further with this approach. If this approach is continued, it goes thus:

$$= 6\ln|\csc\theta-\cot\theta|+C$$

and then we still need to convert $\cot\theta=\frac{1}{\tan\theta}=\frac{1}{\sqrt{x^2-1}}$ and

$$\csc\theta=\sqrt{1+\cot^2\theta}=\sqrt{1+\frac{1}{x^2-1}}=\frac{x}{\sqrt{x^2-1}},$$

hence the integral is

$$= 6\ln\left|\frac{x-1}{\sqrt{x^2-1}}\right| + C \;=\; 6\ln\sqrt{\left|\frac{x-1}{x+1}\right|} + C \;=\; 3\ln\left|\frac{x-1}{x+1}\right| + C.$$

(5 points) $\displaystyle\int \frac{dx}{x^2\sqrt{1-x^2}} =$

Solution. Substitute $x = \sin\theta$, so $\sqrt{1-x^2} = \cos\theta$ and $dx = \cos\theta\, d\theta$ (worth 2 points). The integral becomes

$$\int \frac{\cos\theta d\theta}{\sin^2\theta\cos\theta} = \int \frac{d\theta}{\sin^2\theta} \text{ (worth 2 points)} \quad = \int \csc^2\theta\, d\theta \text{ (worth 3 points)}$$

$$= -\cot\theta + C \text{ (worth 4 points)} \quad = \frac{-\cos\theta}{\sin\theta} + C = \boxed{\frac{-\sqrt{1-x^2}}{x} + C}.$$

(5 points) $\displaystyle\int \frac{dx}{x(x^2+1)} =$

Solution. Write

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Getting that far was worth 2 points (or getting that far but incorrectly was worth 1 point). Then

$$1 = A(x^2+1) + (Bx+C)x$$
$$0x^2 + 0x + 1 = (A+B)x^2 + Cx + A$$

which yields

$$A = 1, \qquad B = -1, \qquad C = 0$$

and now we're up to 3 points. Thus the problem becomes $\displaystyle\int \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx$; getting that far was worth 4 points. Finally, the answer is

$$\boxed{\ln|x| - \frac{1}{2}\ln(x^2+1) + C} \quad \text{or} \quad \boxed{\ln\frac{|x|}{\sqrt{x^2+1}} + C} \quad \text{or} \quad \boxed{\frac{1}{2}\ln\left(\frac{x^2}{x^2+1}\right) + C}.$$

I decided to be lenient about the presence or absence of absolute value signs, but there's really only one correct way to use them — i.e., as I've used them in the boxed answers above.

Some students used a harder, but still viable, method: the identity $1+\tan^2\theta = \sec^2\theta$, with the substitution $x = \tan\theta$ and $\sec\theta = \sqrt{1+x^2}$. Then $dx = \sec^2\theta\, d\theta$. That turns the problem into

$$\int \frac{\sec^2\theta\, d\theta}{\tan\theta \cdot \sec^2\theta} = \int \frac{d\theta}{\tan\theta} = \int \cot\theta\, d\theta \qquad \text{(worth 3 points)}$$

$$= \int \frac{d\sin\theta}{\sin\theta} = \ln|\sin\theta| + C \quad \text{(worth 4 points)}$$

and then finally we have $\sin\theta = \tan\theta/\sec\theta = \frac{x}{\sqrt{1+x^2}}$, which leads to the last boxed answer above.
