Name (please print): **ANSWERS**
Math 150B Quiz 4, Tuesday 10 February 2009, 2 pages, 15 points, 15 minutes

---

(6 points) A spring, when unstretched, has a natural length of 2 meters. If it has already been stretched to a length of 3 meters, maintaining it at that length requires a force of 1 newton. How much work is required to stretch the spring from 2 meters to 3 meters?

Solution. Hooke's Law says that the force needed to maintain a displacement of x is $f(x) = kx$, where k is the spring constant. We're given that a displacement of $3 - 2 = 1$ meter requires a force of 1 newton, so in this problem $k = 1$. Now work equals force times distance, so the work required to move the spring through a small distance dx, from around x to $x + dx$, is $f(x)dx$. We're increasing the displacement from 0 meters to 1 meter, so thus the total work is

$$\int_0^1 f(x)dx = \int_0^1 xdx = \boxed{\frac{1}{2}\text{ joules}} = \boxed{\frac{1}{2}\text{ J}} = \boxed{\frac{1}{2}\text{ newton-meter}}.$$

I deducted one point for omitting the "J" or for writing "N" instead.

One common error was to confuse spring displacement with spring total length, and compute $\int_2^3 f(x)dx = \frac{5}{2}$ joules; I gave 4 points for that (or 3 points for $\frac{5}{2}$N or just $\frac{5}{2}$).

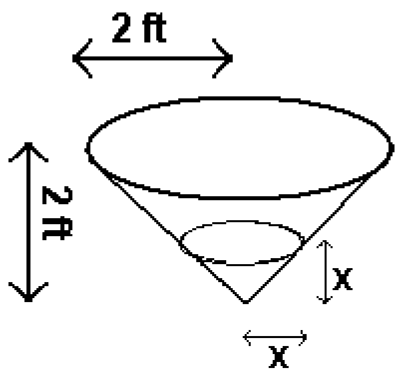
Another common erroneous answer was 1 J, but I gave different amounts of partial credit depending on how that answer was arrived at. Some students used $\int_0^1 f(x)dx$ with $f(x) = kx$, but somehow arrived at $k = 2$; I gave 4 points for that (or 3 points if the "J" was omitted). Other students seemed to think that $f(x)$ was a constant 1; I gave them 3 points (or 2 if the "J" was omitted).

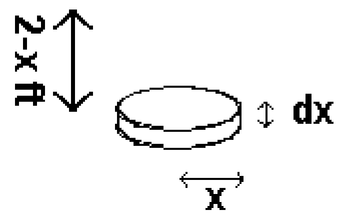
(9 points) Assume that one cubic foot of dirt weighs 100 pounds. How much work is done, in digging a hole that is shaped like a cone whose circular end (in this case the top of the hole, at ground level) has radius 2 feet, and whose pointy end (in this case the bottom of the hole) is 2 feet underground?

(Here "work" refers only to the energy of lifting dirt to the surface level. Ignore the effort of breaking and loosening the soil, and moving excavated dirt away from the hole.)

Solution. We will lift the dirt in circular layers that have thickness dx and that have different radii and depths.

I'm going to let x be the distance between the layer we're considering and the *bottom* of the hole, because it's very slightly easier that way. But later in this explanation, I'll show how to also do it with a different choice of x.





By similar triangles, the layer whose distance is x feet from the bottom of the hole is a circular layer with radius x feet The circle has area πx^2square feet (worth 1 point), and the circular layer has volume $\pi x^2 dx$ cubic feet (worth 2 points). Hence it has weight equal to $100\pi x^2 dx$ pounds (worth 3 points).

(If we were using kilograms — a measure of *mass* — we would next have to multiply by 9.8 m/s^2, the acceleration due to the earth's gravity. But that factor can be omitted, because it is already built into the pounds — a measure not of mass, but of *weight*; weight is a type of *force*. That's why, when we covered this section, we never mentioned what is the number for the acceleration in *feet* per second squared.)

We need to lift it a distance of $2 - x$ feet, so the work involved for that layer is $100\pi x^2(2 - x)dx$ foot-pounds (worth 4 points). The total work is therefore

$$\int_0^2 100\pi x^2(2 - x)dx = 100\pi \int_0^2 (2x^2 - x^3)dx \qquad (\text{either of those is 6 points})$$

$$= 100\pi \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 \qquad (\text{worth 7 points})$$

$$= 100\pi \left[\frac{2 \cdot 8}{3} - \frac{1 \cdot 16}{4}\right] = 100\pi \left[\frac{16}{3} - \frac{12}{3}\right] \qquad (\text{either of those is 8 points})$$

$$= \boxed{\frac{400\pi}{3} \text{ foot-pounds}}.$$

Deduct 1 point for omitting "foot-pounds."

If we had instead let u be the distance between the layer we're considering and the *top* of the hole, we'd end up with $\int_0^2 100\pi(2 - u)^2 u\, du$, which yields the same answer by very slightly more effort.

Common errors:

- I gave 7 points for the incorrect integrals $\int_0^2 100\pi x^3 dx$ or $\int_0^2 100\pi(2-x)^3 dx$ if they were evaluated correctly; they yield an answer of 400π foot-pounds.
- I gave 6 points for the incorrect integrals $\int_0^2 100\pi x^2 dx$ or $\int_0^2 100\pi(2-x)^2 dx$ if they were evaluated correctly; they yield an answer of $800\pi/3$ foot-pounds.
- Many students felt compelled to include a factor of 9.8 (the acceleration due to the earth's gravity, in m/s^2). If you multiply the correct answer by that number, you get $3920\pi/3$ footpounds, for which I gave 7 points. Or, if you divide the correct answer by that number, you get $(400\pi)/(29.4)$ footpounds, for which I gave 6 points. I gave lower scores if those errors were compounded by other errors.