Name (please print): **ANSWERS**
Math 150B Quiz 3, Tuesday 3 February 2009, 2 pages, 15 points, 15 minutes

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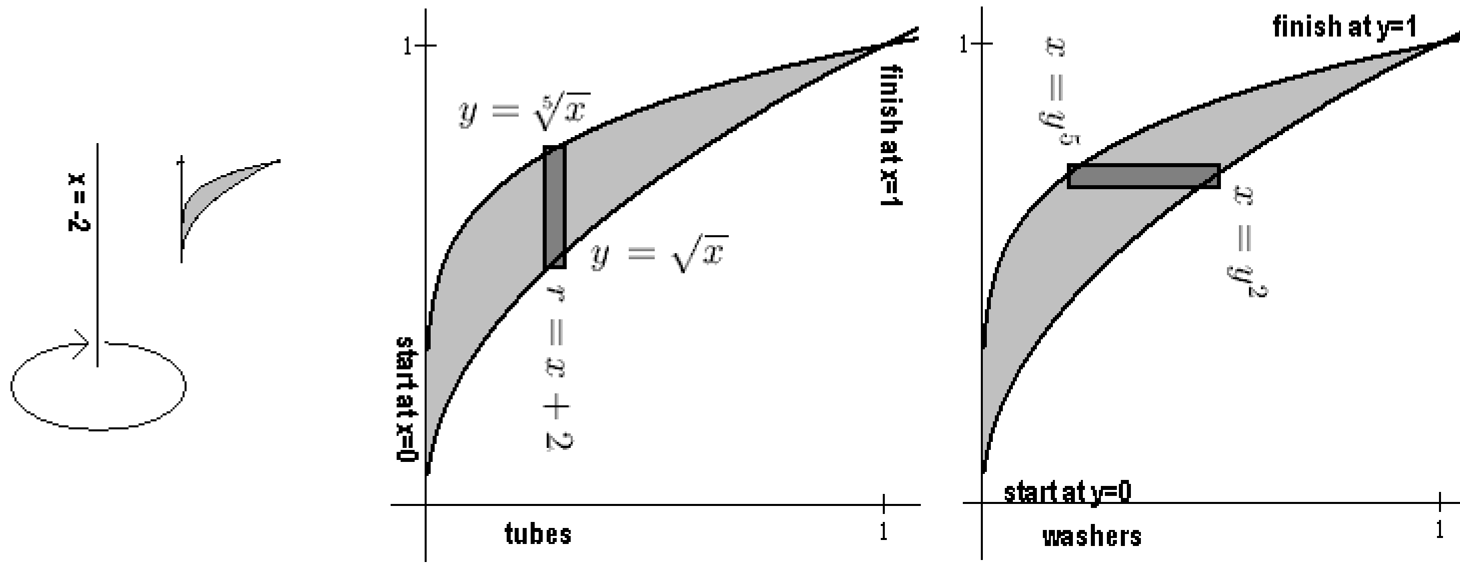
Let R be the region enclosed by the curves $y = \sqrt{x}$ and $y = \sqrt[5]{x}$. **Set up, but do not evaluate**, an integral that gives the volume of:

(4 points) the solid obtained by rotating region R around the line $x = -2$.



Solution. For both this problem and the next one, we need to observe that the intersections of the two curves are at $(0, 0)$ and $(1, 1)$. For the part of the curves between those two points, we have $y = \sqrt{x}$ or $x = y^2$ lower and further to the right than $y = \sqrt[5]{x}$ or $x = y^5$.

Each problem can be done by either washers or tubes.

For the first problem, if we use thin vertical rectangles, then we get tubes. We start at $x = 0$ and finish at $x = 1$. For each value of x, the radius of the tube is $r = x + 2$. The height of the tube is $h = \sqrt[5]{x} - \sqrt{x}$. Thus the volume is $2\pi \int_0^1 rh dx =$

$$\boxed{2\pi \int_0^1 (x + 2)(x^{1/5} - x^{1/2}) dx} = \boxed{2\pi \int_0^1 (2x^{1/5} - 2x^{1/2} + x^{6/5} - x^{3/2}) dx}$$

On the other hand, suppose we use thin horizontal rectangles. Then we go from $y = 0$ to $y = 1$. For each value of y we get a washer, with inner radius equal to $r = 2 + y^5$ and outer radius equal to $R = 2 + y^2$. Then the volume is $\pi \int_0^1 (R^2 - r^2) dy =$

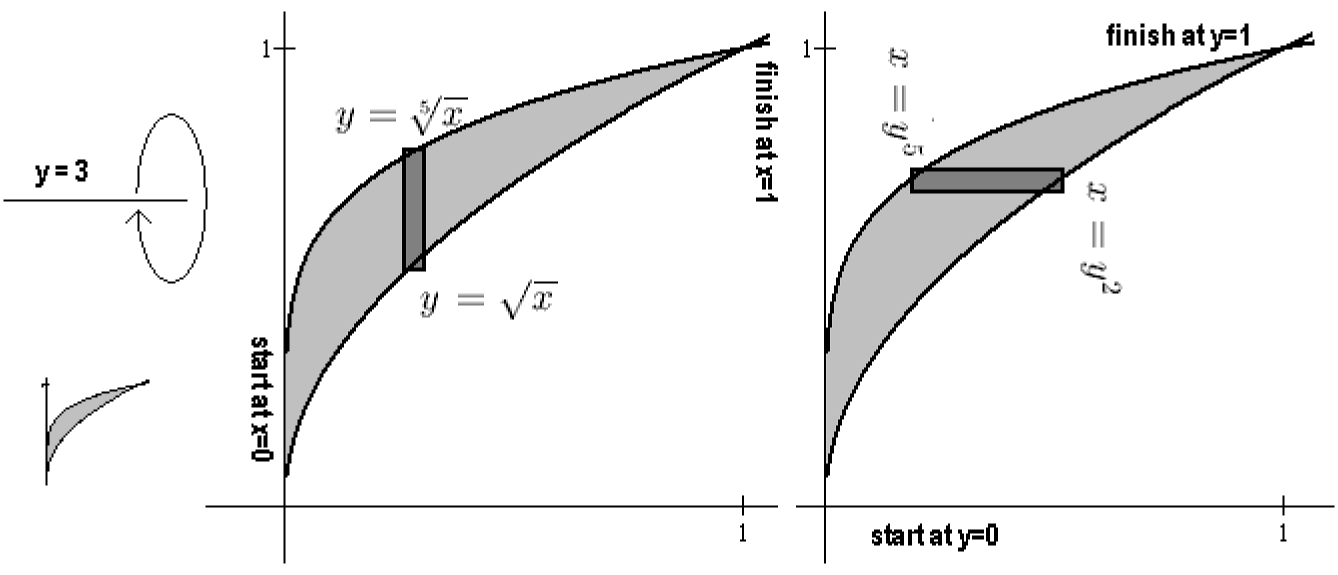
$$\boxed{\pi \int_0^1 \left[(2 + y^2)^2 - (2 + y^5)^2\right] dy} = \boxed{\pi \int_0^1 \left[4y^2 + y^4 - 4y^5 - y^{10}\right] dy}$$

For both methods, one common error was a sign error, for which I deducted 1 point.

Another common error was to rotate region R around the wrong line — e.g., to rotate it around the line $y = -2$. If you use thin vertical rectangles (as in my first picture), then you go from $x = 0$ to $x = 1$, and for each value of x you get a thin washer with inner radius $r = \sqrt{x} + 2$ and outer radius $R = \sqrt[5]{x} + 2$. The resulting volume is then $\pi \int_0^1 (R^2 - r^2) dx = \pi \int_0^1 \left[(\sqrt[5]{x} + 2)^2 - (\sqrt{x} + 2)^2\right] dx$. I gave 3 points for that answer, or less for approximations to that answer (e.g., 2 points for the negative of that answer).

Alternatively, if you go around the line $y = -2$ but use thin horizontal rectangles (as in my second picture), then you go from $y = 0$ to $y = 1$, and for each value of y you get a thin tube with radius $r = y + 2$ and length $h = y^2 - y^5$. That gives volume equal to $2\pi \int_0^1 rh dy = 2\pi \int_0^1 (y+2)(y^2 - y^5) dy$. I gave 3 points for that answer, or less for approximations to that answer (e.g., 2 points for the negative of that answer).

(4 points) the solid obtained by rotating region R around the line $y = 3$.



Solution. For this problem, consider the use of thin vertical rectangles. We go from $x = 0$ to $x = 1$, and for each x we get a thin washer, with inner radius at $r = 3 - \sqrt[5]{x}$ and outer radius at $R = 3 - \sqrt{x}$. Then the volume is $\pi \int_0^1 (R^2 - r^2) dx =$

$$\boxed{\pi \int_0^1 \left[(3 - \sqrt{x})^2 - (3 - \sqrt[5]{x})^2\right] dx} = \boxed{\pi \int_0^1 \left[-6x^{1/2} + x + 6x^{1/5} - x^{2/5}\right] dx}$$

On the other hand, with thin horizontal rectangles, we go from $y = 0$ to $y = 1$, and for each y we get a thin tube, with radius $r = 3 - y$ and width $h = y^2 - y^5$.

Then the volume is $2\pi \int_0^1 rh dy =$

$$\boxed{2\pi \int_0^1 (3-y)(y^2-y^5)dy} = \boxed{2\pi \int_0^1 \left(3y^2 - y^3 - 3y^5 + y^6\right) dy}$$

Again, a sign error cost 1 point.

A common error is to rotate around the line $x = 3$. Using thin vertical rectangles gives thin tubes, $r = 3 - x$, $h = \sqrt[5]{x} - \sqrt{x}$, and volume $2\pi \int_0^1 (3-x)(\sqrt[5]{x} - \sqrt{x})dx$. Using thin horizontal rectangles gives thin washers, $R = 3 - y^5$, $r = 3 - y^2$, volume $\pi \int_0^1 \left[(3-y^5)^2 - (3-y^2)^2\right] dy$. Three points for either of those answers, or less for approximations to those answers.

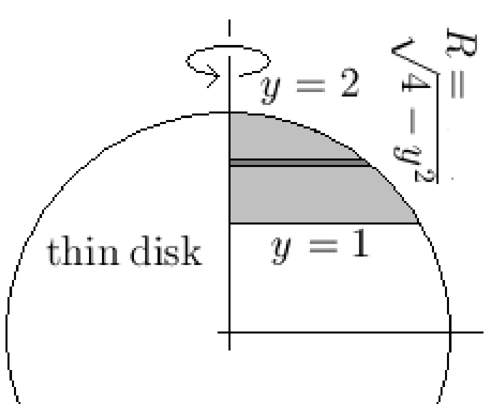
(7 points) Let region R be the region that is inside the circle $x^2 + y^2 = 4$, to the right of $x = 0$, and above $y = 1$. Rotate it around the y-axis. Find the volume of the resulting solid.

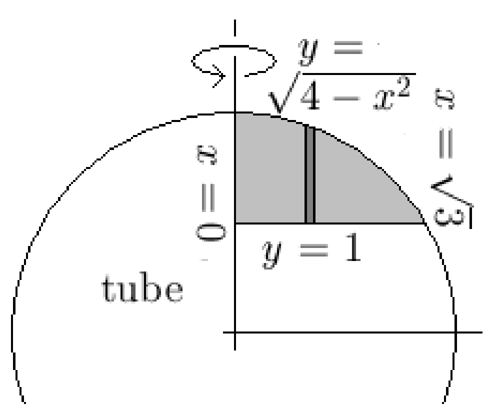
Solution. The given circle is centered at $(0, 0)$ and has radius 2. It intersects the line $y = 1$ at the point $(\sqrt{3}, 1)$.

This problem is simplest with thin horizontal rectangles, which go from $y = 1$ to $y = 2$. For each value of y, we get a thin disk (i.e., inner radius is 0), with outer radius $R = x = \sqrt{4 - y^2}$. The total volume then is $\pi \int_1^2 R^2 dy = \pi \int_1^2 (4 - y^2)dy$. I gave 5 points for getting that far correctly if it was not evaluated correctly; I also gave 5 points for giving an integral differing slightly from that one and then evaluating it correctly. In particular, I gave 5 points for answers of $3\pi\sqrt{3}$ and $11\pi/3$ and $16\pi/3$, when suitable work was shown.

Continuing with the correct evaluation, it is

$$= \pi \left[4y - \frac{1}{3}y^3\right]_1^2 = \pi \left\{4(2-1) - \frac{1}{3}(8-1)\right\} = \pi \left(4 - \frac{7}{3}\right) = \boxed{5\pi/3}\ .$$





The problem can also be done with thin vertical rectangles, which yield tubes, though the problem is then a bit harder. We get one tube for each value of x, as x goes from $x = 0$ to $x = \sqrt{4-1^2} = \sqrt{3}$. Each tube has radius x, and top at $y = \sqrt{4-x^2}$, bottom at $y = 1$, and thus has height equal to $\sqrt{4-x^2} - 1$. Thus the volume is

$$2\pi \int_0^{\sqrt{3}} rh dx \;=\; 2\pi \int_0^{\sqrt{3}} x \left[\sqrt{4-x^2} - 1\right] dx.$$

I gave 5 points for getting that far correctly, if the integral was not evaluated correctly; I also gave 5 points if an integral slightly different from that one was given and then was evaluated correctly.

Continuing with the correct evaluation, it may be easiest to break that integral into two parts:

$$2\pi \int_0^{\sqrt{3}} x\sqrt{4-x^2} dx - 2\pi \int_0^{\sqrt{3}} x dx.$$

That last integral is easy to evaluate: $2\pi \int_0^{\sqrt{3}} x dx = \pi \left[x^2\right]_0^{\sqrt{3}} = 3\pi$. For the other integral, use a substitution, with $u = 4 - x^2$ and $du = -2x dx$. Then

$$2\pi \int_0^{\sqrt{3}} x\sqrt{4-x^2} dx \;=\; -\pi \int_4^1 u^{1/2} du = -\pi \left[\frac{2}{3}u^{3/2}\right]_4^1 = -\frac{2}{3}\pi\,(1-8) = 14\pi/3.$$

Finally, the answer is then $\left(\frac{14}{3} - 3\right)\pi = \boxed{5\pi/3}$,