Name (please print): **ANSWER KEY**
Math 150B Quiz 1, Tuesday 13 January 2009, 2 pages, 15 points, 15 minutes

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CIRCLE YOUR ANSWERS.

(5 points) Find the equation of the line that is tangent to the curve $2\cos(x+y) = x^2 + y^2$ at the point $(1, -1)$.

Solution. Differentiate both sides with respect to x:

$$-2\sin(x+y)(1+y') = 2x + 2yy'.$$

I gave two points for that equation, or any equation equivalent to it.

Now plug in $x = 1$, $y = -1$:

$$0 = 2 - 2y'$$

so $y' = 1$. That's the slope, and the line must pass through $(1, -1)$. Answer: $\boxed{y = x - 2}$. Also full credit for $\boxed{y + 1 = x - 1}$, though I would have preferred the simpler form.

Partial credit. I gave one point for giving an equation of a line — any line — and two points if it was a line that passes through the point $(1, -1)$. Note that an equation of a line is of the form $y = mx + b$ where m and b are *constants*; it's not a line if m and b are functions of x and/or y.

I gave 3 points for an answer of just "1" (the slope of the required line).

I gave 4 points for the equations $y = -x$ or $y + 1 = -(x - 1)$, which are obtained by getting a sign error in the slope of the required line.

I gave 2 points for the equation

$$-2\sin(x+y)(1+y') = 2x + 2yy',$$

or any equation equivalent to it and not substantially more complicated. I believe that I specifically said in class that the easiest way to continue a problem like this one would be to now plug in the given values for x and y, not to solve for y' first. But some students nevertheless solved for y' first. Thus:

$$-\sin(x+y)(1+y') = x + yy',$$

$$-\sin(x+y)y' - yy' = x + \sin(x+y)$$

$$y' = -\frac{x + \sin(x+y)}{y + \sin(x+y)}$$

I gave 3 points for that equation, or any equation equivalent to it and not substantially more complicated. If you get that far, it's still not too late to plug in $x = 1$, $y = -1$, to get $y' = 1$, but some students didn't go for that additional step.

(4 points) Find the derivative of $\sqrt{x + \sin(\sqrt[3]{x})}$.

Solution.

Let $\qquad y = u^{1/2}, \qquad u = x + \sin v, \qquad v = x^{1/3}.$

Then $\qquad \dfrac{dy}{dx} = \dfrac{1}{2}u^{-1/2}\dfrac{du}{dx}, \qquad \dfrac{du}{dx} = 1 + \left[(\cos v)\dfrac{dv}{dx}\right], \qquad \dfrac{dv}{dx} = \dfrac{1}{3}x^{-2/3}.$

Thus $\frac{dy}{dx} = \boxed{\frac{1}{2}\left(x + \sin\sqrt[3]{x}\right)^{-1/2}\left[1 + \frac{1}{3}x^{-2/3}\cos\sqrt[3]{x}\right]}$, which can also be written as any of

$$\boxed{\frac{1 + \frac{1}{3}x^{-2/3}\cos\sqrt[3]{x}}{2\sqrt{x + \sin\sqrt[3]{x}}}} = \boxed{\frac{3 + x^{-2/3}\cos\sqrt[3]{x}}{6\sqrt{x + \sin\sqrt[3]{x}}}} = \boxed{\frac{3\sqrt[3]{x^2} + \cos\sqrt[3]{x}}{6\sqrt[3]{x^2}\sqrt{x + \sin\sqrt[3]{x}}}}.$$

An overwhelmingly popular error was to compute $\frac{du}{dx}$ as $(1 + \cos v)\frac{dv}{dx}$, which leads to the answers

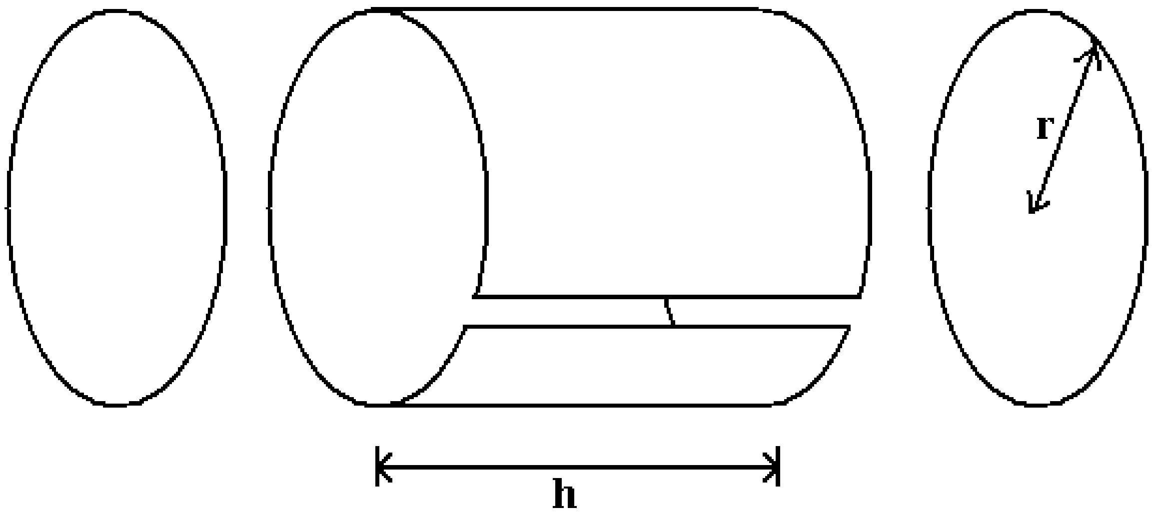
$$\frac{1}{2}\left(x + \sin\sqrt[3]{x}\right)^{-1/2}\left[1 + \cos\sqrt[3]{x}\right]\left(\frac{1}{3}x^{-2/3}\right) \qquad = \qquad \frac{1 + \cos\sqrt[3]{x}}{6x^{2/3}\sqrt{x + \sin\sqrt[3]{x}}}$$

for either of which I gave 3 points (or 2 points for close approximations to those erroneous answers). Just in case the distinction still isn't clear, let me spell it out:

$$1 + \left[\frac{1}{3}x^{-2/3}\cos\sqrt[3]{x}\right] \quad \text{is \textbf{not} equal to} \quad \left[1 + \cos\sqrt[3]{x}\right]\left(\frac{1}{3}x^{-2/3}\right).$$

(6 points) A can is to be made in the usual shape — i.e., what most people call a "cylinder," and what mathematicians call a "right circular cylinder." The side of the can, which is made by curling a rectangle, will be made of a more expensive material; it costs twice as much per square inch as the material used

for the top and bottom flat circles. The total volume of the can is to be 8π cubic inches. What should be the height h and the base radius r, to make the cost as low as possible?



Solution. I have turned the illustration sideways, to better fit it onto this page. You should know from geometry that:

- A circle with radius r has circumference $2\pi r$ and area πr^2.
- A cylinder with base radius r and height h has volume equal to the base area times the height — that is, the volume is $\pi r^2 h$.
- The unrolled rectangle, from the side of the cylinder, has one of its dimensions equal to h, and the other dimension equal to the circle's circumference, $2\pi r$; thus the area of that rectangle is $2\pi r h$.

The surface area consists of the two base circles plus the unrolled side rectangle, so the total surface area is $2\pi r^2 + 2\pi r h$. However, the side rectangle material is twice as expensive, so the cost is

$$\begin{aligned} C &= (\text{area of top}) + (\text{area of bottom}) + 2(\text{area of side}) \\ &= \pi r^2 + \pi r^2 + 2 \cdot 2\pi r h \\ &= 2\pi r^2 + 4\pi r h. \end{aligned}$$

I gave 2 points for that formula or its equivalent, or one point if the surface area $(2\pi r^2 + 2\pi r h)$ was at least given.

We are given that the volume of the can is 8π. But the volume is also equal to $\pi r^2 h$, so we get any of the equations

$$8\pi = \pi r^2 h, \qquad 8 = r^2 h, \qquad h = 8r^{-2}, \qquad r = 2\sqrt{2/h}.$$

I gave 1 point for any of those equations, or their equivalent. Next we must decide which of those last two equations to use. Either will lead to the correct answer, but strategically we should guess which will lead to the correct answer with less effort. I would recommend working with $h = 8r^{-2}$ rather than $r = 2\sqrt{2/h}$, since that way we avoid the messiness of square roots.

Combining the results of the last two paragraphs, we get

$$C(r) = 2\pi r^2 + 32\pi r^{-1}.$$

I gave 4 points for getting that far correctly. We want to make that quantity as small as possible. Compute

$$C'(r) = 4\pi r - 32\pi r^{-2}$$

— getting that far was worth 5 points. Now set $C'(r) = 0$ and solve for r; we get $\boxed{r = 2 \text{ inches}}$, and so $\boxed{h = 2 \text{ inches}}$.

Doing it the hard way: If, instead of using $h = 8r^{-2}$, you used $r = 2\sqrt{2/h} = 2^{3/2}h^{-1/2}$, the subsequent computation is

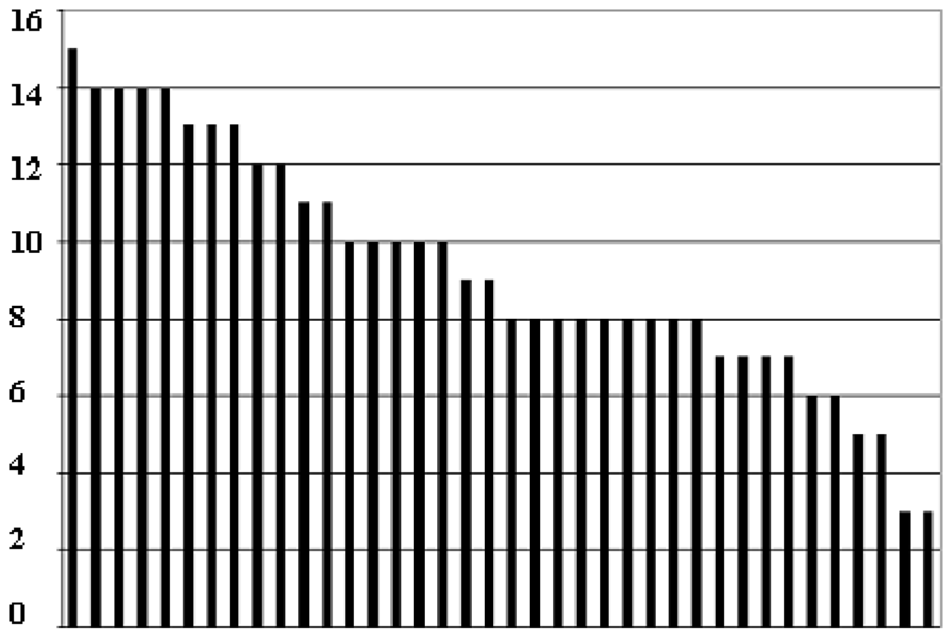
$$\begin{array}{rcll} C(h) & = & 16\pi h^{-1} + 2^{7/2}\pi h^{1/2} & \text{(worth 4 points), and then} \\ C'(h) & = & -16\pi h^{-2} + 2^{5/2}\pi h^{-1/2} & \text{(worth 5 points).} \end{array}$$

Setting $C'(h)$ equal to 0 and solving for h yields $h = 2$ and therefore $r = 2$.

Common errors: Several students set the volume equal to the surface area — that is, $8\pi = 2\pi r^2 + 2\pi rh$. I have no idea why, and did not give any partial credit for that, but I did give one point for giving the formula for the surface area correctly.

Very few students got full credit on this problem. Many students seemed to be clueless; I don't really know whether that was because they don't know how to do word problems or because they ran out of time.

Out of 38 students who took this quiz, one had a perfect score. Class average: 9.26 points out of 15, which is 61.75 percent, or a grade of D minus. Evidently I overestimated what this class is capable of; probably my subsequent quizzes won't be quite this hard. The distribution of scores is shown below.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