n-parameter families of curves

For purposes of this discussion, a **curve** will mean any equation involving x, y, and no other variables. Some examples of curves are

$x^2 + (y-3)^2 = 9$	circle with radius 3, centered at $(0, 3)$
$x^2 = y$	parabola
$x = y^2$	another parabola
$x^3 e^y + \sin(xy) = 2\ln(x^2 + y^2 + 1)$	I don't know what this looks like, but it's still a curve

Some curves can obviously be viewed as functions. For instance,

- $y = x^2$ obviously makes y into a function of x, so dy/dx makes sense anywhere along that curve. We say that y is expressed **explicitly** as a function of x; that means we have a formula of the form
$$y \quad = \quad \begin{pmatrix} \text{some expression} \\ \text{not involving } y \end{pmatrix}.$$
- $x = y^2$ makes x into a function of y, so dx/dy makes sense anywhere along that curve. We say that x is expressed **explicitly** as a function of y; that means we have a formula of the form
$$x \quad = \quad \begin{pmatrix} \text{some expression} \\ \text{not involving } x \end{pmatrix}.$$

The circle $x^2 + (y-3)^2 = 9$ satisfies neither of those conditions — it is neither explicit for y in x, nor explicit for x in y. We say that the relation between x and y is represented **implicitly**.

However, even the circle gives us an explicit relation *locally, at most points.* For instance,

> the upper half-circle $\Big\{(x, y) \; : \; x^2 + (y-3)^2 = 9, \; y > 3\Big\}$ can be rewritten as $y = 3 + \sqrt{9 - x^2}$, making y into a function of x, so dy/dx makes sense. (Remember that the symbol $\sqrt{\ }$ means "the nonnegative square root of," so it always takes a value greater than or equal to zero. To make it two-valued you have to put $\pm$ in front of it.)

In a similar fashion, the lower half circle gives y as a function of x, where dy/dx also makes sense; and the right or left half circle gives x as a function of y, where dx/dy makes sense. Thus both derivatives are defined everywhere on the circle except at four points. That's enough for most purposes of this course, which is primarily concerned with the **local behavior** of functions — i.e., what happens near a point. (Global behavior is considered in more advanced courses.)

In a similar fashion, on "most" curves (with exceptions such as a vertical line or a horizontal line), the derivatives are defined at "most" points (with exceptions at corners, local maxima or minima, etc.). Rather than thinking of one of x, y as a "function" of the other, it may be more helpful to think of x and y as two **related** quantities: When one changes, then the other changes in a corresponding fashion. The derivatives dy/dx and dx/dy give the relative rates of those changes.

Many curves can also be expressed **parametrically** — i.e., with x and y both given as functions of a third variable. For instance, the circle $x^2 + (y-3)^2 = 9$ (given by one equation in two variables) can also be expressed as

$$x(\theta) = 3\cos\theta, \qquad y(\theta) = 3 + 3\sin\theta \qquad (0 \le \theta \le 2\pi)$$

(two equations in 3 variables). But we won't be using that kind of parametrization in this discussion (except once near the very end when we consider Clairaut's equation). The discussion in this document will be largely concerned with another, very different use of the word "parameter":

A **one-parameter family of curves** is the collection of curves we get by taking an equation involving x, y, and one other variable — for instance, c (though any other letter will do just as well). Plugging in different numbers for c gives us different curves; in many cases those curves are related in some way that is visually simple. Here are some examples:

- The equation $x^2 + (y - c)^2 = 9$ represents a one-parameter family of curves. That is, it represents infinitely many different curves. Indeed, it represents the collection of all circles that have radius 3 and that are centered at points along the line $x = 0$. A few of the curves it represents are

$c = 3$	$x^2 + (y - 3)^2 = 9$	circle centered at (0,3)
$c = -1$	$x^2 + (y + 1)^2 = 9$	circle centered at $(0, -1)$
$c = \pi + \sqrt{17}$	$x^2 + (y - \pi - \sqrt{17})^2 = 9$	circle centered at $(0, \pi + \sqrt{17})$

  For comparison, note that this family can also be written as $x^2 + (y - c)^2 - 9 = 0$.

- More generally, if $f(x, y)$ is some function of two variables, then $f(x, y - c) = 0$ is a one-parameter family of curves. To get the graph of one of these curves from the graph of another, just translate (move) it vertically.

  Similarly, $f(x + b, y) = 0$ represents a one-parameter family; its curves can be obtained from each other by horizontal translation.

- The equation $y = ax^2$ represents infinitely many functions of x. For each value of a, we get y equal to one function of x. The curves represented are the parabolas with axis of symmetry along the y-axis and with vertex at $(0, 0)$. They differ from one another in that they are stretched vertically by a factor of a (or stretched horizontally by $\sqrt{a}$).

Similarly, a **two-parameter family of curves** is the collection of curves we get by taking an equation involving x, y, and two other variables, *provided that that family of curves cannot also be represented using just one parameter.* That last clause is a bit subtle, and will be illustrated by the third example below. Some examples:

- The equation $(x-a)^2+(y-b)^2 = 9$ is a 2-parameter family of curves. It represents the collection of all circles of radius 3. Plugging in some number for a and some number for b gives us one particular circle of radius 3, the one centered at (a, b). Note that the numbers a and b can be chosen *independently* of each other. We can rewrite the equation as $(x - a)^2 + (y - b)^2 - 9 = 0$.

  More generally, the equation $f(x - a, y - b) = 0$ represents the set of all translates (horizontal and vertical combined) of the curve $f(x, y) = 0$.

- The equation $y = ax + b$ represents all straight lines. This is a 2-parameter family.

- At first glance, the equation $y = ax + bx$ might look like a 2-parameter family of curves to beginners. But it is actually a 1-parameter family of curves; it simply has not been written in the most economical form.

Indeed, rewrite it as $y = (a + b)x$. You'll see that it is a straight line, passing through $(0, 0)$, with slope equal to $a + b$. The same straight line is represented by $y = mx$ if we simply take $m = a + b$. Here are some of the curves that belong to this family of curves:

a	b	curve	m
3	2	$y = 5x$	5
1	7	$y = 8x$	8
12	-7	$y = 5x$	5
12	-4	$y = 8x$	8

Choosing $a = 3$ and $b = 2$ yields the *same curve* as choosing $a = 12$ and $b = -7$; both those curves are represented by $y = mx$ with $m = 5$. In other words,

> the set of curves that we can get from $y = ax + bx$ by plugging in some number for a and some number for b

is the same as

> the set of curves that we can get from $y = mx$ by plugging in some number for m.

More generally, for any positive integer n, an n-**parameter family of curves** is the collection of curves we get by taking an equation involving x, y, and n other variables, provided that that family of curves cannot be represented with fewer parameters.

Once you've understood the definitions, it should be fairly obvious that $y = ax^2 + bx + c$ is a 3-parameter family while $y = ax + bx$ is only a 1-parameter family. But some examples are not so obvious. For instance,

$$(*) \qquad y = a\sin^2 x + b\cos^2 x + c$$

looks like a 3-parameter family of curves, because the functions $\sin^2 x$ and $\cos^2 x$ and 1 (the constant function) are different functions. But we can rewrite this equation as

$$y = (a - b)\sin^2 x + (b + c).$$

The curves we can get by plugging particular numbers in for a, b, c are the same as the curves we can get from

$$y = p\sin^2 x + q$$

by plugging in particular numbers for p and q. Thus, $(*)$ is actually a 2-parameter family of curves.

Exercises on families of curves. Represent each of the following as an n-parameter family of curves — i.e., represent it by an equation in x and y with n additional variables, where n is as low as possible. Simplify as much as you can; then circle your final answer.

(A) All horizontal lines.

(B) All circles in the plane. (*Hint*: Translations and size.)

(C) With y equal to a function of x: All functions that are polynomials of degree 3 or less.

(D) All parabolas that have vertical lines for their axes of symmetry. (*Hint*: Translations and vertical stretching, or think in terms of polynomials.)

(E) All ellipses whose axes of symmetry are parallel to the x- and y- coordinate axes. (*Hint*: Start with a circle; apply translations, horizontal stretching, and vertical stretching.)

Making Differential Equations

A **differential equation** is an equation involving derivatives. The **order** of a differential equation is the highest number of derivations that have been applied to any term in the equation. For instance,

$$\frac{d^3y}{dx^3} + 3x^3\frac{d^2y}{dx^2} + 2xy\frac{dy}{dx} + 16e^x y = 0$$

is a third-order differential equation, because it involves $\dfrac{d^3y}{dx^3}$ but does not involve any higher derivatives.

(This should not be confused with the *degree* of a differential equation. That's harder to define, but here's an example: The equation $y^2\dfrac{dy}{dx} = \sin^3 x$ could be said to be of third degree, because the term $y^2(dy/dx)^1$ has exponents summing to 3.)

To *solve* a differential equation means, roughly, to find the set of all the functions which satisfy that differential equation. It turns out that, for the most part,

the general solution of an nth order differential equation is an n-parameter family of curves.

(We'll come back to that "for the most part" business later, in the last section of this document.) Most of this course is concerned with solving differential equations — i.e.,

Given a differential equation, find the solution.

But, just to help us master the concept of "solution," we'll start with this **reverse problem**, which is much easier:

Given the solution, find the corresponding differential equation.

The procedure is quite simple to describe (though in some problems the the actual computations get messy):

1. Algebraically solve for one of the parameters. For instance, if one of the parameters is a, rewrite the equation so that it is in the form

$$a = \begin{pmatrix} \text{some expression} \\ \text{not involving } a \end{pmatrix}.$$

(If there is more than one parameter, it doesn't matter which one you start with; pick whichever one seems most convenient.) Actually, you could instead solve for something of the form

$$\left(\begin{array}{c}\text{some expression involving}\\ a \text{ and no other letters}\end{array}\right) = \left(\begin{array}{c}\text{some expression}\\ \text{not involving } a\end{array}\right).$$

The left side of that equation might be $1/a$, or $2a^2$, or $3a^3+\sin(a)$, or something like that. Any equation of this sort will do just as well, and one of these variants may be more convenient than another — i.e., it might allow us a simpler choice on the right side of the equation, and that will be advantageous in subsequent steps.

2. Differentiate both sides of the equation with respect to x. (Or with respect to some other variable, if that is more convenient.) Keep in mind that y is a function of x, so its derivative is y'. Use the chain rule wherever needed. The derivative of any constant is 0. Thus, we get

$$0 = \left(\begin{array}{c}\text{some new expression}\\ \text{not involving } a\end{array}\right).$$

We have eliminated a from the equation. But we have also raised the order of the differential equation by one.

3. Now pick another parameter — say b — and repeat the process. Continue until all the parameters are gone. Each repetition reduces the number of parameters by one, and raises the order of the differential equation by one; thus an n-parameter solution yields an nth-order differential equation.

4. Simplify the result as much as possible.

Following are some examples.

An example with parabolas. Find the differential equation whose solution is $y = ax^2$.

Answer. Rewrite as $a = x^{-2}y$. Differentiate both sides with respect to x, viewing y as a function of x. On the left side we'll just get 0. On the right side, we have to use the rule for the derivative of a product of two functions:

$$\begin{aligned}\frac{d}{dx}\left(x^{-2}\cdot y\right) &= \left(\frac{d}{dx}x^{-2}\right)y + x^{-2}\left(\frac{d}{dx}y\right)\\ &= -2x^{-3}y + x^{-2}y'.\end{aligned}$$

Thus we get the differential equation $0 = -2x^{-3}y + x^{-2}y'$. Simplify, to

$$\boxed{xy' = 2y} \quad \text{or} \quad \boxed{\frac{dy}{dx} = 2\frac{y}{x}}.$$

Checking the answer. If $y = ax^2$, then $y' = 2ax$. Do those satisfy the answer we arrived at? I.e., do we have $xy' \stackrel{?}{=} 2y$? Well, that equation simplifies to $x \cdot 2ax \stackrel{?}{=} 2 \cdot ax^2$, which is indeed true.

An example with some bizarre curve. $3x^2y + x\cos(xy) + e^x \ln y = c$. I have no idea what this curve looks like, but its differential equation is not hard to find.

Answer. It's already solved for c; all we have to do is differentiate both sides. That's going to confuse some students, who need to review "implicit differentiation" from calculus. I'll break up into little steps for you:

$$3x^2y + x\cos(xy) + e^x \ln y = c$$

Differentiate both sides;

$$\frac{d}{dx}\left(3x^2y\right) + \frac{d}{dx}\left(x\cos(xy)\right) + \frac{d}{dx}\left(e^x \ln y\right) = 0.$$

Use the product rule, $(uv)' = u'v + uv'$. Thus

$$\left[\frac{d}{dx}\left(3x^2\right)\right] y + 3x^2\left[\frac{d}{dx}y\right] + \left[\frac{dx}{dx}\right]\cos(xy) + x\left[\frac{d}{dx}\cos(xy)\right] + \left[\frac{d}{dx}e^x\right]\ln y + e^x\left[\frac{d}{dx}\ln y\right] = 0.$$

That simplifies a little, to

$$6xy + 3x^2y' + \cos(xy) + x\left[\frac{d}{dx}\cos(xy)\right] + e^x\ln y + e^x\left[\frac{d}{dx}\ln y\right] = 0.$$

We're going to need the chain rule. It helps to think of y as a function of x; write it as $y(x)$ if you like.

$$\frac{d\ln y}{dx} = \frac{d\ln y}{dy}\cdot\frac{dy}{dx} = \frac{1}{y}\cdot y'.$$

Similarly, xy is a function of x; it could be written as $x\cdot y(x)$ if you like. Using the chain rule again, and also the product rule:

$$\frac{d\cos(xy)}{dx} = \frac{d\cos(xy)}{d(xy)}\cdot\frac{d(xy)}{dx} = [-\sin(xy)]\cdot\left[\frac{dx}{dx}\cdot y + x\cdot\frac{dy}{dx}\right] = -\sin(xy)(y + xy').$$

Substituting those results into our differential equation yields

$$\boxed{6xy + 3x^2y' + \cos(xy) - x\sin(xy)(y + xy') + e^x\ln y + e^xy^{-1}y' = 0}.$$

If you want to emphasize the differential equation aspect of this, you could group all the y' terms together: $\boxed{6xy + \cos(xy) - xy\sin(xy) + e^x\ln y + \left[3x^2y' - x^2\sin(xy) + e^xy^{-1}\right]y' = 0}$. Or, moving the differentials to opposite sides of the equation,

$$\boxed{\left[xy\sin(xy) - 6xy - \cos(xy) - e^x\ln y\right]dx \;=\; \left[3x^2y' - x^2\sin(xy) + e^xy^{-1}\right]dy}.$$

An example: Ellipses or hyperbolas with axes on the coordinate axes. That can be written as

$$ax^2 + by^2 = 1. \tag{1}$$

What is the differential equation?

Answer. Let's begin by solving for a; we get

$$a = x^{-2}(1 - by^2). \tag{1'}$$

Now comes our first differentiation. We want to differentiate both sides with respect to x. Here a and b are constants, and we will view y as a function of x. Here are some facts from calculus that we will need. (You may need to review your calculus.)

$$\begin{array}{rcccc}
\frac{d}{dx}(a) & = & 0 & & \\
\frac{d}{dx}\left(x^{-2}\right) & = & -2x^{-3} & & \\
\frac{d}{dx}(uv) & = & u\frac{dv}{dx} & + & v\frac{du}{dx} \\
\frac{d}{dx}\left(y^2\right) & = & 2y\frac{dy}{dx} & = & 2yy' \\
\frac{d}{dx}\left(x^{-2}y^2\right) & = & x^{-2}\frac{d}{dx}\left(y^2\right) & + & y^2\frac{d}{dx}\left(x^{-2}\right) \\
 & = & 2x^{-2}yy' & - & 2x^{-3}y^2
\end{array}$$

Thus, differentiating equation (1′) yields

(2) $$0 = -2x^{-3}(1 - by^2) + x^{-2}(-2byy').$$

To simplify a little, multiply through by $-x^3/2$; we obtain

(2′) $$0 = 1 - by^2 + bxyy'.$$

Solving for b yields

(2″) $$b = \frac{1}{y^2 - xyy'}.$$

If we differentiate both sides of that, the left side will get replaced by 0, but the right side will get messy. Here's a trick to save some work: Before differentiating, modify both sides so that the differentiation will be easier. In this example, that can be accomplished by taking the reciprocal on both sides. Thus we obtain

(2‴) $$\frac{1}{b} = y^2 - xyy'.$$

Now differentiate both sides. The left side still gets replaced by 0, but we save a lot of work on differentiating the right side. The result is

(3) $$0 = 2yy' - yy' - xy'y' - xyy''.$$

Simplify to $\boxed{yy' = xy'y' + xyy''}$. This is a second-order differential equation, since it involves y''. But checking this one is messy.

An example: more hyperbolas. Find the differential equation for $(x - a)(y - b) = 4$.

Answer. We can write y as an explicit function of x:

$$y = b + \frac{4}{x - a}.$$

Differentiating once gets rid of b; we have $y' = -4(x-a)^{-2}$. (Note that y' is less than 0; the function $b + \frac{4}{x-a}$ is decreasing on each of the two intervals where it is defined.) Now solve that equation algebraically for a; we get

$$a = x \pm (-\frac{1}{4}y')^{-1/2}.$$

Differentiating both sides of that yields

$$0 = 1 \pm \left(\frac{-1}{2}\right)\left(\frac{-1}{4}y'\right)^{-3/2}\left(\frac{-1}{4}y''\right).$$

Simplifying,

$$\left(\frac{-1}{4}y'\right)^{3/2} = \pm\frac{1}{8}y''$$

Square both sides and simplify; $\boxed{(y'')^2 + (y')^3 = 0}$.

Checking: $y' = -4(x-a)^{-2}$ and $y'' = 8(x-a)^{-3}$, hence $0 \stackrel{?}{=} (y'')^2 + (y')^3 = [8(x-a)^{-3}]^2 + [-4(x-a)^{-2}]^3 = (64-64)(x-a)^{-6} \stackrel{\checkmark}{=} 0$.

Exercises on making differential equations. Find the differential equations for these families of curves. Simplify; then circle your final answer. (You are urged to check your answers, too.)

(F) Straight lines with slope 2/3. These have equation $y = \frac{2}{3}x + b$. You should get a first-order differential equation. *Optional hint:* If you look ahead to the next section, you'll see that this one-parameter family is additive with a nonzero particular term, so the resulting differential equation should be linear nonhomgeneous.

(G) All straight lines. That's $y = mx + b$. Eliminate both m and b, to obtain a second-order differential equation. *Optional hint:* This is additive with two homogeneous terms and no particular term, so the differential equation should be linear homogeneous.

(H) The circles tangent to the x-axis at $(0,0)$ have equation $x^2 + (y-b)^2 = b^2$. Find the corresponding first-order differential equation. *Optional hint*: Not additive; answer is not linear.

(I) $y = ax^2 + bx^{-1/3}$. *Optional hint*: Linear homogeneous.

(J) $y = a\sin x + b\cos x$. *Hint*: Instead of solving for a or b and then differentiating, use this alternate method, which works well for this problem but not for many other problems: Differentiate the given equation to obtain formulas for y' and y''. Thus we have three equations:

$$y = a\sin x + b\cos x, \qquad y' = [\text{what}], \qquad y'' = [\text{what}],$$

where the right sides involve a, b, x, but not y. Then algebraically eliminate a and b from those three equations, to find one second-order differential equation involving some or all of x, y, y', y'' but not explicitly mentioning a or b.

(K) $y = a\sin\sqrt{x} + b\cos\sqrt{x}$. Hint: Similar to previous problem, but there's more computation. Don't forget to use the chain rule.

Envelope Solutions

(It might be best to postpone this topic for a while — e.g., until one has covered separation of variables and other elementary methods for solving first order differential equations.)

I stated earlier that, **for the most part**, the solution of an nth order differential equation is an n-parameter family of curves. That's exactly right when we work with *linear* equations — our topic for most of the semester — but we get exceptions with some nonlinear differential equations. The next few pages consider some of the simplest exceptions. (I will postpone until later in the semester a discussion of what "linear" means in this context.)

Theorem on envelopes. Let $G(x, y.c)$ be a function of three variables. Thus the equation $G(x, y, c) = 0$ represents a one-parameter family of curves, if we view c as a parameter. Let us denote by $\mathcal{E}_c$ the curve given by parameter value c.

Suppose that that one parameter family of curves satisfies some first-order differential equation. Then there may also be an additional curve satisfying that differential equation, which is tangent to each $\mathcal{E}_c$ at one point. This curve envelopes the $\mathcal{E}_c$'s, so it is called their **envelope**. (In some books it is also called a **singular solution** to the differential equation.)

If an envelope solution exists, it can be found by this procedure:

$$(*) \qquad G(x, y, c) = 0 \qquad \text{and} \qquad \frac{\partial G}{\partial c}(x, y, c) = 0$$

is a system of two equations in the three variables x, y, c. By algebraic means, we may eliminate c, thus obtaining one equation in just the variables x and y. That equation is the envelope curve.

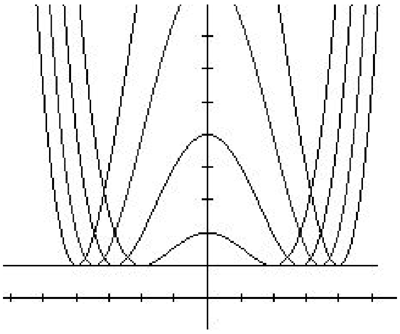
The proof uses the vector version of the chain rule. We won't go into any of the details of that, but we will give several examples.

Example 1. The nonlinear differential equation $\dfrac{dy}{dx} = x\sqrt{y-1}$ can be solved by the method of separation of variables, discussed later in this course. The general one-parameter solution turns out to be $y = 1+(\frac{1}{4}x^2+c)^2$, a family of fourth-degree polynomials. We can rewrite that as $G(x, y, c) = 0$, if we use the function $G(x, y, c) = 1 + (\frac{1}{4}x^2 + c)^2 - y$.

Now compute $\dfrac{\partial G}{\partial c} = \dfrac{1}{2}x^2 + 2c$. We want to eliminate c from the system of two equations $(*)$ mentioned in the theorem — that is, the system of these two equations:

$$1 + \left(\frac{1}{4}x^2 + c\right)^2 - y = 0 \qquad \text{and} \qquad \frac{1}{2}x^2 + 2c = 0.$$

Probably the easiest way to do that is to solve the second equation for c; we get $c = -\frac{1}{4}x^2$. Now plug that in for c in the first equation. We end up with the horizontal line $y = 1$ for the envelope curve in this example.



The curves $y = 1 + (\frac{1}{4}x^2 + c)^2$ make up a one-parameter family of solutions to the differential equation $y' = x\sqrt{y-1}$. The envelope of those curves is y = 1; it also satisfies that differential equation.

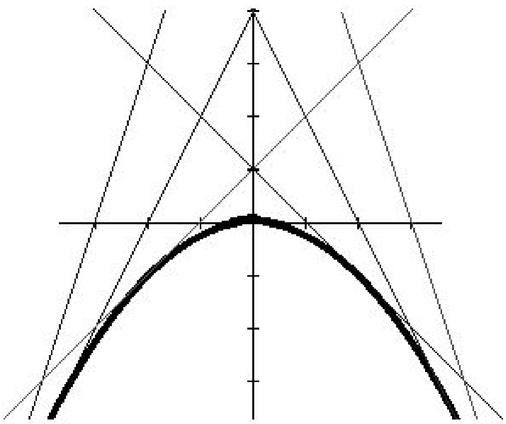
Example 2. The nonlinear differential equation $y = xy' + (y')^2$ can be shown to have one parameter family of solutions $y = cx + c^2$, a family of straight lines. (Those c's are the same. If you prefer to see the c only once, write the family instead as $y = (c + \frac{1}{2}x)^2 - \frac{1}{4}x^2$. But that would make the computations below more complicated.)

We can represent the solution family as $G(x, y, c) = 0$ if we use the function $G(x, y, c) = cx + c^2 - y$.

Now compute $\frac{\partial G}{\partial c} = x + 2c$. We want to eliminate c from the system of two equations $(*)$ mentioned in the theorem — that is, the system of these two equations:

$$cx + c^2 - y = 0 \qquad \text{and} \qquad x + 2c = 0.$$

The second of those equations yields $c = -\frac{1}{2}x$. Plug that into the first equation; thus we obtain $(-\frac{1}{2}x)x + (-\frac{1}{2}x)^2 - y = 0$. That simplifies to $y = -x^2/4$, a parabola.



The differential equation $y = x\,y' + (y')^2$ has a one-parameter family of solutions consisting of the straight lines $y = cx + c^2$. Those lines have envelope given by the parabola $y = -x^2/4$, which also satisfies the differential equation.

Actually, we do have to choose G with some skill. For instance, we could have represented that one-parameter family of straight line solutions as $y = (c + \frac{1}{2}x)^2 - \frac{1}{4}x^2$, or even as $\sqrt{y + \frac{1}{4}x^2} = c + \frac{1}{2}x$. Thus, we could have used the function $G(x, y, c) = \sqrt{y + \frac{1}{4}x^2} - c - \frac{1}{2}x$. But that yields $\partial G/\partial c = -1$, and so the second equation in the system $(*)$ is $-1 = 0$. Thus, there is no point (x, y) in the plane that satisfy satisfies both equations of $(*)$, since no point satisfies the second equation. This approach does not yield the envelope solution — i.e., it yields the wrong answer. [I haven't yet figured out how to avoid this difficulty, except to recommend that you sketch a graph when possible to see if there might be an envelope solution. Watch for a later edition of this document.]

Example 3. The preceding example is a particular instance (obtained with $h(c) = c^2$) of an entire broad class of examples:

Let h be *any* continuously differentiable function of one variable (other than a constant function). Then the equation $y = xy' + h(y')$ is known as **Clairaut's equation**. It's not hard to verify that the straight lines $y = cx + h(c)$ form a one-parameter solution to Clairaut's equation. After a bit of computation, we arrive at the envelope solution, which can be expressed parametrically:

$$x(t) = -h'(t), \qquad\qquad y(t) = -th'(t) + h(t).$$

Example 4. The differential equation $xdy + ydx = 0$ has general solution $x^2 + y^2 = c$. That's the set of all circles centered at 0 — a family of concentric circles. Sketching the graph (not shown here) makes it obvious that there are no envelope solutions. The theorem on envelopes agrees with that conclusion — it tells us to look for a simultaneous solution of the two equations

$$x^2 + y^2 = c \qquad \text{and} \qquad -1 = 0.$$

Obviously those have no simultaneous solution — i.e., there are no points (x, y) in the plane that satisfy both these equations. In this case, that's correct, as we can see by looking at the graph.

Exercises on evelopes. Find the envelope for each of the following families of curves. (You don't need to find the differential equation.)

(L) $y = e^{-x}\sin(x + c)$. *Hint:* The envelope consists of two curves.

(M) $y = \sin(\frac{1}{2}x^2 + c)$. *Hint:* The envelope consists of two straight lines.

(N) $(x - c)(y - c) = -1$. *Hint:* The envelope consists of two straight lines.

(O) $y - x = 2(c - x)(1 - c)$.

This problem has a history for me personally: When I was a child, one of the toys I played with was a square frame lined with evenly spaced pegs, over which one could loop some colored elastic bands. One of my favorite patterns consisted of the arrangement shown in the accompanying figure. The straight line for parameter c passes through the points $(c - 1, 1 - c)$ and (c, c); that gives us the one-parameter family indicated above. I could see a curve being formed, but I didn't know much geometry back then. The only curves I knew well were circles, so I guessed the envelope curve was a quarter of a circle. (I didn't know it was called an envelope.) Actually, it turns out to be a parabola, not a circle. You just have to find the parabola's equation.