(4 points) Find all values of $z$ satisfying $z^2 + 4z + 11 = 0$.

**Solution.** Completing the square yields $(z + 2)^2 = z^2 + 4z + 4 = -7$, hence $z = \left[ -2 \pm i\sqrt{7} \right]$. We can also obtain that answer using the quadratic formula, with $a = 1$, $b = 4$, $c = 11$:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{-28}}{2} = \frac{-4 \pm 2\sqrt{-7}}{2} = \left[ -2 \pm i\sqrt{7} \right] .$$

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(5 points) $y''' + y'' - 6y' = 0$

**Solution.** This is linear homogeneous with constant coefficients. The associated polynomial equation is $k^3 + k^2 - 6k = 0$. That factors as $k(k - 2)(k + 3)$, which has the three distinct real roots 0, 2, -3. Thus the answer is $y = c_1 e^{0x} + c_2 e^{2x} + c_3 e^{-3x}$, or more simply $y = c_1 + c_2 e^{2x} + c_3 e^{-3x}$.

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(8 points) $\frac{dy}{dx} - 2xy = xy^2$

**Solution.** This problem is Bernoulli with $n = 2$. We begin to solve it by substituting $u = y^{1-n} = y^{-1}$, that yields $u' = -y^{-2}y'$. Multiply the given equation through by $-y^{-2}$ to obtain

$$-y^{-2}y' + 2xy^{-1} = -x$$

which now can be rewritten as

$$u' + 2xu = -x.$$ 

That’s linear in standard form. It has $P(x) = 2x$, hence $\int P(x)dx = x^2$, hence integrating factor $I(x) = e^{x^2}$. Multiply that onto both sides of the linear standard form equation, to obtain

$$e^{x^2}u' + 2xe^{x^2}u = -xe^{x^2}$$
That is,
\[
(e^{x^2} u)' = -xe^{x^2}.
\]
Integrate both sides;
\[
e^{x^2} u = -\int xe^{x^2} dx = -\frac{1}{2} \int e^{x^2} d(x^2) = -\frac{1}{2} e^{x^2} + C.
\]
Divide out the \(e^{x^2}\); thus \(u = -\frac{1}{2} + Ce^{-x^2}\). That is,
\[
y^{-1} = -\frac{1}{2} + Ce^{-x^2}.
\]
I gave full credit if you stopped there, but I would have preferred that you take one more step:
\[
y = \frac{1}{-\frac{1}{2} + Ce^{-x^2}} \quad \text{or} \quad (\text{with a different } C) \quad y = \frac{2}{-1 + Ce^{-x^2}}.
\]
(Some students took that last step incorrectly, thinking that \(\frac{1}{p+q}\) is equal to \(\frac{1}{p} + \frac{1}{q}\). But those are \textbf{not} equal. They ended up with \(y\) equal to \(-2 + Ce^{x^2}\). That’s a serious conceptual error, so I deducted 2 points for it.)

Check: If \(y^{-1} = -\frac{1}{2} + Ce^{-x^2}\), then \(e^{x^2} y^{-1} = -\frac{1}{2} e^{x^2} + C\). Differentiating both sides yields \(e^{x^2} (2xy^{-1} - y^{-2} y') = -xe^{x^2}\). Multiply through by \(y^2 e^{-x^2}\) to obtain
\[
2xy - y' = -xy^2 \quad \checkmark.
\]

(8 points) The general solution of
\[
2x^2 y'' - xy' + y = x^4
\]
is of the form \(y = c_1 y_1 + c_2 y_2 + y_p\), where \(c_1\) and \(c_2\) are arbitrary constants. Given the hint that
\[
y_1 = x,
\]
find the general solution (i.e., find \(y_2\) and \(y_p\)). I’d recommend using the method of reduction of order, but you may use another method if you know one.

\textit{Solution.} Substitute \(y = uy_1 = ux\). That yields \(y' = u'x + u\) and \(y'' = u''x + 2u'\). Plug those into the given equation, to obtain
\[
2x^2(u''x + 2u') - x(u'x + u) + ux = x^4
\]
The $ux$ terms cancel out, and the $u'x^2$ terms combine. That simplifies to

$$2x^3u'' + 3x^2u' = x^4$$

Substitute $u' = v$, to obtain

$$2x^3v' + 3x^2v = x^4$$

which is a first order linear equation; 4 points for getting that far correctly. Divide out $2x^3$, to put it in standard form:

$$v' + \frac{3}{2}x^{-1}v = \frac{1}{2}x$$

Thus we have $P(x) = \frac{3}{2}x^{-1}$ and $\int P(x)dx = \frac{3}{2}\int x^{-1}dx = \frac{3}{2}\ln x$ and $I(x) = e^{\int P} = e^{(3/2)\ln x} = x^{3/2}$. That’s the integrating factor. Multiply it onto both sides of the standard form equation;

$$x^{3/2}v' + \frac{3}{2}x^{1/2}v = \frac{1}{2}x^{5/2}$$

Now we’re up to 6 points. Integrate both sides;

$$x^{3/2}v = \frac{1}{2} \cdot \frac{2}{7}x^{7/2} + C_1 = \frac{1}{7}x^{7/2} + C_1$$

$$u' = v = \frac{1}{7}x^2 + C_1x^{-3/2}$$

Integrate both sides again:

$$u = \frac{1}{21}x^3 + B_1x^{-1/2} + B_2$$

Finally, $y = ux$ yields

$$y = \frac{1}{21}x^4 + B_1x^{1/2} + B_2x$$

or

$$y_2 = x^{1/2}$$

and $y_p = \frac{1}{21}x^4$.

Checking:

If

$$y = \frac{1}{21}x^4 + B_1x^{1/2} + B_2x$$

then

$$y' = \frac{4}{21}x^3 + \frac{1}{2}B_1x^{-1/2} + B_2$$

$$y'' = \frac{12}{21}x^2 - \frac{1}{4}B_1x^{-3/2}$$

$$2x^2y'' - xy' + y = \frac{21}{21}x^4 + 0B_1x^{1/2} + 0B_2x$$

$$= \sqrt{1}$$

$$= -x$$

$$= 2x^2$$