Name (please print):
Math 170 Test 1, 5 Feb 2010, 4 pages, 25 points, 50 minutes.

(6 points) Find the second order differential equation whose general solution is $y = ax^2 + b$. Simplify your answer as much as possible.

Solution. First, $b$ is already isolated; differentiate both sides with respect to $x$, to eliminate $b$:

$$y' = 2ax$$

Next, to isolate $a$, multiply both sides by $x^{-1}$:

$$x^{-1}y' = 2a.$$  

Then differentiate both sides with respect to $x$:

$$x^{-1}y'' - x^{-2}y' = 0.$$  

Finally, multiply through by $x^2$ to simplify:

$$xy'' - y' = 0$$ or $$xy'' = y'$$

which can also be written in a few other ways. You can check your result this way:

If $y = ax^2 + b$ then $y' = 2ax$  
$y'' = 2a$  
$xy'' = 2ax$

and so $xy'' = y'$.

(7 points) Find the centroid of the region that is enclosed by the curves $y = x^2$ and $y = x$. (Hint: First find the points where those two curves intersect.)

Solution. The intersections occur at $x^2 = x$, i.e., at $x^2 - x = 0$; that is, $x(x - 1) = 0$. That’s at $x = 0$ and at $x = 1$.

The simpler analysis is to express everything in terms of the variable $x$. Chop the region into thin vertical rectangles. The rectangle whose horizontal coordinate is $x$ has top at $y = x$ and bottom at $y = x^2$, so its height is $x - x^2$, its width
is $dx$, its area is $(x - x^2)dx$, and its center is at $(x, \frac{x+x^2}{2})$. Its moment is equal to its center times its area; that is,

$$
\left(x(x - x^2)dx, \frac{x + x^2}{2}(x - x^2)dx\right) = \left((x^2 - x^3)dx, \frac{1}{2}(x^2 - x^4)dx\right).
$$

To find the moment of the system, add the moments of the rectangles — that is, integrate the above expression from $x = 0$ to $x = 1$. Thus we get

$$
\left(\int_0^1 (x^2 - x^3)dx, \frac{1}{2}\int_0^1 (x^2 - x^4)dx\right) = \left(\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1, \frac{1}{2}\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1\right)
$$

$$
= \left(\frac{1}{3} - \frac{1}{4}, \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)\right) = \left(\frac{1}{12}, \frac{1}{15}\right).
$$

To find the total mass (or area) of the system, add the areas of all the rectangles — that is, integrate

$$
\int_0^1 (x - x^2)dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.
$$

Finally, the centroid of the system is its total moment divided by its total mass — that is,

$$
\left(\frac{1}{12}, \frac{1}{15}\right) \div \frac{1}{6} = \left(\frac{1}{12}, \frac{1}{15}\right) \cdot 6 = \left(\frac{1}{2}, \frac{2}{5}\right) = (0.5, 0.4).
$$

This problem can also be done in terms of $y$, but it’s harder that way. For that approach, use thin horizontal rectangles. The typical rectangle has right end at $x = \sqrt{y}$ and right end at $x = \sqrt{y}$, so it has width $\sqrt{y} - \sqrt{y}$, area $(\sqrt{y} - \sqrt{y})dy$, and center at $(\frac{\sqrt{y} + \sqrt{y}}{2}, y)$. And so on.
(6 points) Solve \( \frac{dy}{dx} = \frac{(2x - 1)y}{(x^2 + 1)(2y^2 - 1)}. \)

**Solution:** Separate the variables:

\[
\frac{2y^2 - 1}{y} \, dy = \frac{2x - 1}{x^2 + 1} \, dx
\]

Integrate both sides:

\[
\int \frac{2y^2 - 1}{y} \, dy = \int \frac{2x - 1}{x^2 + 1} \, dx
\]

\[
\left(2y - y^{-1}\right) \, dy = \int \left(\frac{2x}{x^2 + 1} - \frac{1}{x^2 + 1}\right) \, dx
\]

\[
y^2 - \ln|y| = \ln(x^2 + 1) - \arctan(x) + C
\]

which cannot be solved explicitly for \( x \) or for \( y \). But it can be written in a few other forms, which some might deem simpler:

\[
y^2 = \ln|\(x^2 + 1\)| - \arctan x + C
\]

\[
x = \tan\left(\ln|\(x^2 + 1\)| - y^2 + C\right)
\]

I charged a point for omitting the \(+ C\), or for not remembering the arctangent integral correctly. I did not deduct any points for omitting the absolute value bars around \( y \) (though really they ought to be present), nor for including absolute value bars around \( x^2 + 1 \) (though really they ought to be absent). Neither absolute value is needed in this formulation of the answer:

\[
(x^2 + 1)y = ke^{y^2 - \arctan x}
\]

because the \( \pm \) can be absorbed into the \( k = \pm e^{-C} \).

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(6 points) Solve \( x \frac{dy}{dx} + 2y = x^2 + x. \)

**Solution.** Put it in standard form, by dividing through by \( x \):

\[
\frac{dy}{dx} + \frac{2}{x} y = x + 1
\]
Thus we identify \( P(x) = \frac{2}{x} \), so \( \int P(x)dx = \int \frac{2}{x}dx = 2\ln x \). Then the integrating factor is \( e^{\int P} = e^{2\ln x} = x^2 \). Multiply that onto both sides of the standard form equation:

\[
x^2 \frac{dy}{dx} + 2xy = x^3 + x^2
\]

\[
(x^2y)' = x^3 + x^2
\]

Integrate both sides:

\[
x^2y = \int (x^3 + x^2)dx = \frac{1}{4}x^4 + \frac{1}{3}x^3 + C
\]

Solve for \( y \):

\[
y = \frac{1}{4}x^2 + \frac{1}{3}x + Cx^{-2}.
\]

Total number of points is 25.