Name (please print):
Math 170 Test 1, 5 Feb 2010, 4 pages, 25 points, 50 minutes.

(6 points) Find the second order differential equation whose general solution is $y = ax^2 + b$. Simplify your answer as much as possible.

Solution. First, b is already isolated; differentiate both sides with respect to x, to eliminate b:

$$y' = 2ax$$

Next, to isolate a, multiply both sides by x^{-1}:

$$x^{-1}y' = 2a.$$

Then differentiate both sides with respect to x:

$$x^{-1}y'' - x^{-2}y' = 0.$$

Finally, multiply through by x^2 to simplify:

$$\boxed{xy'' - y' = 0} \quad \text{or} \quad \boxed{xy'' = y'}$$

which can also be written in a few other ways. You can check your result this way:

$$\begin{array}{rrcl} \text{If} & y & = & ax^2 \ +b \\ \text{then} & y' & = & 2ax \\ & y'' & = & 2a \\ & xy'' & = & 2ax \end{array}$$

and so $xy'' = y'$.

(7 points) Find the centroid of the region that is enclosed by the curves $y = x^2$ and $y = x$. (*Hint*: First find the points where those two curves intersect.)

Solution. The intersections occur at $x^2 = x$, i.e., at $x^2 - x = 0$; that is, $x(x-1) = 0$. That's at $x = 0$ and at $x = 1$.

The simpler analysis is to express everything in terms of the variable x. Chop the region into thin vertical rectangles. The rectangle whose horizontal coordinate is x has top at $y = x$ and bottom at $y = x^2$, so its height is $x - x^2$, its width is dx, its area is $(x - x^2)dx$, and its center is at $\left(x,\ \frac{x+x^2}{2}\right)$. Its moment is equal to its center times its area; that is,

$$\left(x(x - x^2)dx,\ \frac{x + x^2}{2}(x - x^2)dx\right) = \left((x^2 - x^3)dx,\ \frac{1}{2}(x^2 - x^4)dx\right).$$

To find the moment of the system, add the moments of the rectangles — that is, integrate the above expression from $x = 0$ to $x = 1$. Thus we get

$$\left(\int_0^1 (x^2 - x^3)dx,\ \frac{1}{2}\int_0^1 (x^2 - x^4)dx\right) = \left(\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1,\ \frac{1}{2}\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1\right)$$

$$= \left(\frac{1}{3} - \frac{1}{4},\ \frac{1}{2}(\frac{1}{3} - \frac{1}{5})\right) = \left(\frac{1}{12},\ \frac{1}{15}\right).$$

To find the total mass (or area) of the system, add the areas of all the rectangles — that is, integrate

$$\int_0^1 (x - x^2)dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Finally, the centroid of the system is its total moment divided by its total mass — that is,

$$\left(\frac{1}{12},\ \frac{1}{15}\right) \div \frac{1}{6} = \left(\frac{1}{12},\ \frac{1}{15}\right) \cdot 6 = \boxed{\left(\frac{1}{2},\ \frac{2}{5}\right)} = \boxed{(0.5,\ 0.4)}.$$

This problem can also be done in terms of y, but it's harder that way. For that approach, use thin horizontal rectangles. The typical rectangle has right end at $x = \sqrt[3]{y}$ and right end at $x = \sqrt{y}$, so it has width $\sqrt[3]{y} - \sqrt{y}$, area $(\sqrt[3]{y} - \sqrt{y})dy$, and center at $\left(\frac{\sqrt[3]{y}+\sqrt{y}}{2},\ y\right)$. And so on.

(6 points) Solve $\dfrac{dy}{dx} = \dfrac{(2x-1)y}{(x^2+1)(2y^2-1)}$.

Solution: Separate the variables:

$$\frac{2y^2-1}{y}dy = \frac{2x-1}{x^2+1}dx$$

Integrate both sides:

$$\int \frac{2y^2-1}{y}dy = \int \frac{2x-1}{x^2+1}dx$$

$$\int \left(2y - y^{-1}\right) dy = \int \left(\frac{2x}{x^2+1} - \frac{1}{x^2+1}\right) dx$$

$$\boxed{y^2 - \ln|y| = \ln(x^2+1) - \arctan(x) + C}$$

which cannot be solved explicitly for x or for y. But it can be written in a few other forms, which some might deem simpler:

$$\boxed{y^2 = \ln\left|(x^2+1)y\right| - \arctan x + C}$$

$$\boxed{x = \tan\left(\ln\left|(x^2+1)y\right| - y^2 + C\right)}$$

I charged a point for omitting the $+C$, or for not remembering the arctangent integral correctly. I did not deduct any points for omitting the absolute value bars around y (though really they ought to be present), nor for including absolute value bars around x^2+1 (though really they ought to be absent). Neither absolute value is needed in this formulation of the answer:

$$\boxed{(x^2+1)y = ke^{y^2+\arctan x}}$$

because the $\pm$ can be absorbed into the $k = \pm e^{-C}$.

(6 points) Solve $x\dfrac{dy}{dx} + 2y = x^2 + x$.

Solution. Put it in standard form, by dividing through by x:

$$\frac{dy}{dx} + \frac{2}{x}y = x + 1$$

Thus we identify $P(x) = \frac{2}{x}$, so $\int P(x)dx = \int \frac{2}{x}dx = 2\ln x$. Then the integrating factor is $e^{\int P} = e^{2\ln x} = x^2$. Multiply that onto both sides of the standard form equation:

$$\begin{aligned} x^2\frac{dy}{dx} + 2xy &= x^3 + x^2 \\ \left(x^2 y\right)' &= x^3 + x^2 \end{aligned}$$

Integrate both sides:

$$x^2 y = \int (x^3 + x^2)dx = \frac{1}{4}x^4 + \frac{1}{3}x^3 + C$$

Solve for y:

$$\boxed{y = \tfrac{1}{4}x^2 + \tfrac{1}{3}x + Cx^{-2}}.$$

Total number of points is 25.