Name (please print):
Math 170 Test 3, Fri 9 April 2010, 3 pages, 25 points, 50 minutes.

Find the sum of the series:

(5 points) $3 - 2 + \frac{4}{3} - \frac{8}{9} + \frac{16}{27} - \frac{32}{81} + \frac{64}{243} - \cdots = \boxed{}$

Solution. This is a geometric series; see page 725. We have

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots \; = \; \frac{a}{1-r} \quad \text{if } |r| < 1.$$

In the present problem, we have $a = 3$ and $r = -2/3$, so the sum is

$$\frac{a}{1-r} = \frac{3}{1 - -(2/3)} = \frac{3}{1 + (2/3)} = \frac{3}{5/3} = \boxed{\frac{9}{5}} = \boxed{1.8}.$$

(4 points) $\ln\frac{3}{4} + \ln\frac{8}{9} + \ln\frac{15}{16} + \ln\frac{24}{25} + \ln\frac{48}{49} + \cdots = \boxed{}$ *Hint*: This series can be rewritten as $\sum_{k=2}^{\infty}\left[\ln\left(\frac{k-1}{k}\right) - \ln\left(\frac{k}{k+1}\right)\right]$.

Solution. This is a telescoping series. If we stop after the nth term, we get

$$s_n = \sum_{k=2}^{n}\left[\ln\left(\frac{k-1}{k}\right) - \ln\left(\frac{k}{k+1}\right)\right]$$

$$= \ln\left(\frac{2-1}{2}\right) - \ln\left(\frac{n}{n+1}\right) = \ln\frac{1}{2} - \ln\left(\frac{n}{n+1}\right)$$

which converges to $\boxed{\ln(1/2)}$ or $\boxed{-\ln 2}$ or $\boxed{-0.693\ldots}$. Or, another way to analyze this problem is:

$$s = \left(\ln\frac{1}{2} - \ln\frac{2}{3}\right) + \left(\ln\frac{2}{3} - \ln\frac{3}{4}\right) + \left(\ln\frac{3}{4} - \ln\frac{4}{5}\right) + \left(\ln\frac{4}{5} - \ln\frac{5}{6}\right) + \cdots$$

Every term except that first $\ln \frac{1}{2}$ cancels out; and if you stop with one term on the end uncanceled, that last uncanceled term is $-\ln\left(\frac{k}{k+1}\right)$, which is converging to $-\ln(1) = 0$.

For each of the following series, circle either the word "convergent" or the word "divergent" (whichever it is).

(4 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3 - 7n + \sqrt{n}}{n^9 + 6n^2 + 2}$ convergent divergent

Solution. $\boxed{\text{convergent}}$ by the limit comparison test: Take

$$a_n = \frac{n^3 - 7n + \sqrt{n}}{n^9 + 6n^2 + 2} \qquad \text{and} \qquad b_n = \frac{n^3}{n^9} = \frac{1}{n^6}.$$

Then it is easy to see that $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, and that $\sum b_n$ is convergent by the p-series test with $p = 6$.

(4 points) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ convergent divergent

Solution. Note that this does not say $\sin(n\pi)$ or $\sin(n\pi/2)$ or anything like that. The numbers $\sin(n)$ behave erratically — sometimes positive, sometimes negative, not alternating, not in a simple pattern. However, they always satisfy $|\sin(n)| \leq 1$. So reason in this fashion: The series $\displaystyle\sum \frac{1}{n^2}$ converges by the p-series test (with $p = 2$), and we have

$$0 \leq \frac{|\sin(n)|}{n^2} \leq \frac{1}{n^2}$$

so the series $\displaystyle\sum \frac{|\sin(n)|}{n^2}$ is convergent by the comparison test. Therefore the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ is absolutely convergent, and hence it is $\boxed{\text{convergent}}$.

(4 points) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ convergent divergent

Solution. Use the integral test. In $\displaystyle\int_2^\infty \frac{dx}{x\,(\ln x)^3}$, substitute $u = \ln x$ and $du = \frac{dx}{x}$. The integral becomes $\int_{\ln 2}^{\infty} u^{-3} du$, which is $\boxed{\text{convergent}}$.

(4 points) The radius of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{2^n x^n}{n^2}$ is $R = \boxed{}$.

Solution. This is probably done most easily with the ratio test: $c_n = \frac{2^n}{n^2}$ and $c_{n+1} = \frac{2^{n+1}}{(n+1)^2}$, so

$$\frac{c_n}{c_{n+1}} = \frac{(n+1)^2}{2n^2} = \frac{1}{2}\left(\frac{n+1}{n}\right)^2 \text{ which converges to } \boxed{\frac{1}{2}}.$$

Partial credit: 2 points for an answer of 2.