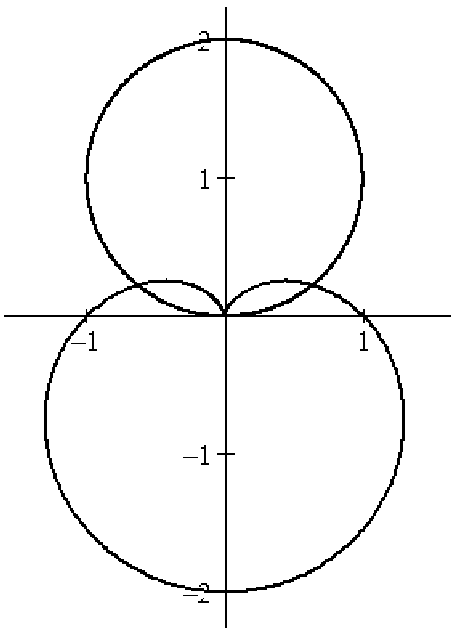
Name (please print):
Math 170 Test 2, 5 March 2010, 4 pages, 25 points, 50 minutes.

(6 points) Find all three points of intersection of the cardioid $r = 1 - \sin\theta$ and the circle $r = \sin\theta$. You may express your answer in polar or Cartesian coordinates.



Solution. Setting the two r's equal to each other yields

$$1 - \sin\theta = \sin\theta$$
$$1 = 2\sin\theta$$
$$\sin\theta = 1/2$$
$$\theta = \pi/6 \quad \text{and} \quad \theta = 5\pi/6.$$

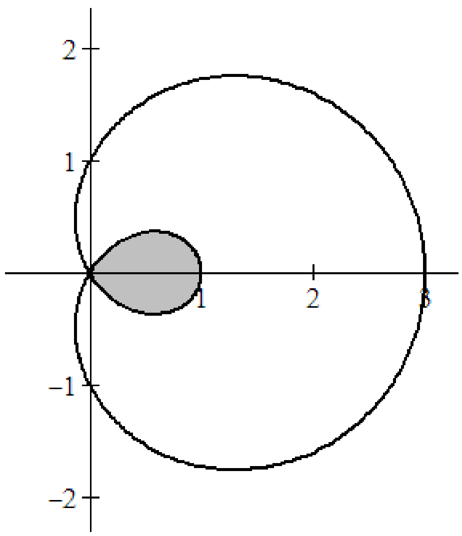
Inspection shows that the third intersection is at the origin. Thus the answer can be expressed either in polar coordinates:

$$\boxed{(r,\theta) = (0,\ \text{any}),\ \left(\frac{1}{2},\frac{\pi}{6}\right) \approx (0.5, 0.524),\ \left(\frac{1}{2},\frac{5\pi}{6}\right) \approx (0.5, 2.618)}$$

or in Cartesian coordinates:

$$\boxed{(x,y) = (0,0),\ \left(\frac{\pm 1}{4}\sqrt{3},\ \frac{1}{4}\right) \approx (\pm 0.433, 0.25)}\ .$$

(8 points) The limaçon $r = 1 + 2\cos\theta$ consists of a little loop inside a big loop. Find the area of just the little loop. *Hint*: The curve crosses over itself — transitioning between big and little loop — at the origin, but you'll have to figure out what is the value of θ there. *Another hint*: $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$.



Solution. The curve goes through the origin at $r = 0$, so $\cos\theta = -1/2$, which is at $\theta = 2\pi/3$ and $\theta = 4\pi/3$. Thus we have $0 \le \theta \le 2\pi/3$ on the outer loop, $2\pi/3 \le \theta \le 4\pi/3$ on the inner loop, and $4\pi/3 \le \theta \le 2\pi$ back on the outer loop. The area, then, is

$$\int_{2\pi/3}^{4\pi/3} \frac{1}{2} r^2 \, d\theta \;=\; \int_{2\pi/3}^{4\pi/3} \frac{1}{2}(1 + 2\cos\theta)^2 d\theta$$

— getting that far is worth 5 points. Continuing,

$$= \int_{2\pi/3}^{4\pi/3} \left[\frac{1}{2} + 2\cos\theta + 2\cos^2\theta\right] d\theta \;=\; \int_{2\pi/3}^{4\pi/3} \left[\frac{1}{2} + 2\cos\theta + 1 + \cos 2\theta\right] d\theta$$

(worth 6 points)

$$= \left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{2}\sin 2\theta\right]_{2\pi/3}^{4\pi/3}$$

(worth 7 points)

$$= \frac{3}{2}\left(\frac{4\pi}{3} - \frac{2\pi}{3}\right) + 2\left(\sin\frac{4\pi}{3} - \sin\frac{2\pi}{3}\right) + \frac{1}{2}\left(\sin\frac{8\pi}{3} - \sin\frac{4\pi}{3}\right)$$

$$= \frac{3}{2}\left(\frac{2\pi}{3}\right)+2\left[\left(-\frac{1}{2}\sqrt{3}\right)-\left(\frac{1}{2}\sqrt{3}\right)\right]+\frac{1}{2}\left[\left(\frac{1}{2}\sqrt{3}\right)-\left(-\frac{1}{2}\sqrt{3}\right)\right] = \boxed{\pi-\tfrac{3}{2}\sqrt{3}}$$

which is approximately equal to $\boxed{0.5435}$.

The computation could be made a little easier by symmetry: take just the first half of the little loop, and then multiply the result by two. The computation then is

$$\int_{2\pi/3}^{\pi}(1+2\cos\theta)^2 d\theta \qquad \text{(worth 5 points)}$$

$$= \int_{2\pi/3}^{\pi}[3+4\cos\theta+2\cos 2\theta]\,d\theta \qquad \text{(worth 6 points)}$$

$$= [3\theta+4\sin\theta+\sin 2\theta]_{2\pi/3}^{\pi} \quad \text{(7 points)} \qquad = \pi-\frac{3}{2}\sqrt{3}.$$

(8 points) In the interval $0 \le \theta \le 2\pi$, which values of θ yield a point on the spiral $r = e^{-\theta}$ where the tangent to the curve is a horizontal line? a vertical line?

Horizontal at $\theta =$		Vertical at $\theta =$	

Solution. We have

$$\begin{aligned} x &= r\cos\theta = e^{-\theta}\cos\theta & \tfrac{dx}{d\theta} &= e^{-\theta}(-\cos\theta-\sin\theta) \\ y &= r\sin\theta = e^{-\theta}\sin\theta & \tfrac{dy}{d\theta} &= e^{-\theta}(\cos\theta-\sin\theta) \end{aligned}$$

and finally $\frac{dy}{dx} = \left(\frac{dy}{d\theta}\right) / \left(\frac{dx}{d\theta}\right)$.

We get a horizontal tangent wherever $\frac{dy}{dx} = 0$; that is wherever $\frac{dy}{d\theta} = 0$. That gives us $\cos\theta - \sin\theta = 0$, and so

$$\boxed{\text{horizontal at } \theta = \pi/4 \text{ and at } \theta = 5\pi/4}.$$

We get a vertical tangent whereever $\frac{dx}{d\theta} = 0$; that gives us $-\cos\theta - \sin\theta = 0$, and so

$$\boxed{\text{vertical at } \theta = 3\pi/4 \text{ and at } \theta = 7\pi/4}.$$

(3 points) Translate the equation $r = \cos\theta - \sin\theta$ into Cartesian coordinates (that is, x, y). Also, describe the curve in words. A sketch is not required.

Solution. Multiply both sides of the given equation by r, to obtain $r^2 = r\cos\theta - r\sin\theta$. That translates to

$$\boxed{x^2 + y^2 = x - y}.$$

That equation can be rearranged to

$$x^2 - x + \frac{1}{4} + y^2 + y + \frac{1}{4} = \frac{1}{2}$$

$$\boxed{\left(x - \tfrac{1}{2}\right)^2 + \left(y + \tfrac{1}{2}\right)^2 = \left(\tfrac{1}{\sqrt{2}}\right)^2.}$$

Thus it is

$$\boxed{\begin{array}{c}\text{a circle centered at } \left(\tfrac{1}{2}, -\tfrac{1}{2}\right) \\ \text{with radius } 1/\sqrt{2}\end{array}}$$

but I'll give full credit for just describing it as $\boxed{\text{a circle}}$.

Total number of points is 25.