Name: **ANSWER KEY**

Math 170 Quiz 3, 26 Mar 2010, 1 page, 15 points, 20 minutes.

(3 points) Convergent or divergent? $\sum_{n=1}^{\infty}(-1)^n \frac{n+3}{n+2}$

Solution. $\boxed{\text{Divergent}}$, since $\frac{n+3}{n+2}$ does not converge to 0. I gave one point of partial credit to anyone who asserted anything about $\lim_{n\to\infty} \frac{n+3}{n+2}$, since at least they were asking some of the right questions.

(4 points) Convergent or divergent? $\sum_{n=1}^{\infty} \frac{n^2+3}{\sqrt{n^7+2n+5}}$

Solution. $\boxed{\text{Convergent}}$. Analyze this as follows: Let

$$a_n = \frac{n^2+3}{\sqrt{n^7+2n+5}} \qquad \text{and} \qquad b_n = \frac{n^2}{\sqrt{n^7}} = \frac{1}{n^{3/2}}.$$

It's easy to prove (or to see, without any written computation, by the methods of textbook section 4.4) that $\lim_{n\to\infty} a_n/b_n = 1$. And then $\sum b_n$ behaves the same as $\int_1^\infty \frac{dx}{x^{3/2}}$, which is convergent. In general, we have

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} 1/(p-1) & \text{if } p > 1, \\ \text{divergent} & \text{if } p \le 1, \end{cases}$$

and

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{convergent} & \text{if } p > 1, \\ \text{divergent} & \text{if } p \le 1, \end{cases}$$

though we don't have an easy way to calculate particular values of $\zeta(p)$. (*A bit of trivia*: It turns out that $\zeta(2) = \pi^2/6$, but that's hard to prove.)

Some students did not understand the correct use of the limit comparison test. What it says is that if a_n/b_n converges to a finite positive limit, then $\sum a_n$ and $\sum b_n$ both converge or both diverge. But some students chose some b_n that yields $a_n/b_n \to 0$ or $a_n/b_n \to \infty$, and they drew some conclusion from that, and in this they were mistaken. Conceptual error; zero partial credit. If a_n/b_n does *not* converge to a finite positive limit, then the limit comparison test is inconclusive, because you've made the wrong choice of b_n, and you need to try again with a different choice of b_n.

(4 points) Find the sum of the series $\quad 3 - \frac{9}{4} + \frac{27}{16} - \frac{81}{64} + \frac{243}{256} - \frac{729}{1024} + \cdots$

Solution. This is $3\left[1 + \left(-\frac{3}{4}\right) + \left(-\frac{3}{4}\right)^2 + \left(-\frac{3}{4}\right)^3 + \cdots\right]$ or $3\sum_{n=0}^{\infty}\left(-\frac{3}{4}\right)^n$ — i.e., it is 3 times a geometric series. (I gave 1 point for getting that far.) We know that $1 + x + x^2 + x^3 + x^4 + \cdots = 1/(1-x)$ if $|x| < 1$. (Another point for writing something like that.) In this case $x = -3/4$. So we get

$$\frac{1}{1-x} = \frac{1}{1 - -\frac{3}{4}} = \frac{1}{\frac{7}{4}} = \frac{4}{7}$$

and then finally the series is 3 times that, so the answer is $\boxed{12/7}$, or $\boxed{1.714\ldots}$. Another way to answer this problem is to know the formula

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r} \qquad \text{if } |r| < 1;$$

in this case $a = 3$ and $r = -3/4$.

A common error was to use $r = 3/4$ instead; that leads to an answer of $\frac{3}{1-\frac{3}{4}} = \frac{3}{1/4} = 12$, for which I gave 3 points.

(4 points) Find the sum of $\quad \sum_{n=2}^{\infty} \frac{2}{n^2-1} = \frac{2}{3} + \frac{2}{8} + \frac{2}{15} + \frac{2}{24} + \frac{2}{35} + \frac{2}{48} + \cdots$

Solution. Two points for observing any of the following: It's a telescoping series, or it can be written as $\sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$, or it can be written as

$$\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots$$

For full credit, observe that the only terms that don't cancel out against later terms are $\frac{1}{1} + \frac{1}{2}$, and so the sum is $\boxed{3/2} = \boxed{1.5}$.
