Name: **ANSWER KEY**.
Math 170 Quiz 2, 19 Feb 2010, 1 page, 15 points, 20 minutes.

All three of the problems below are concerned with the curve

$$x = e^{t/2}\cos t, \qquad y = e^{t/2}\sin t, \qquad 0 \le t \le \pi/2.$$

To save you some time, I will do part of the computation for you:

$$x'(t) = e^{t/2}\left(\frac{1}{2}\cos t - \sin t\right), \qquad y'(t) = e^{t/2}\left(\frac{1}{2}\sin t + \cos t\right).$$

(6 points) Find the arclength of the curve. *Hint*: If you don't make any mistakes, the expression under the square root sign will simplify.

Solution.

$$\begin{aligned} & (x')^2 + (y')^2 \\ = \;& e^t\left[\tfrac{1}{4}\cos^2 t - \cos t \sin t + \sin^2 t\right] + e^t\left[\tfrac{1}{4}\sin^2 t + \sin t\cos t + \cos^2 t\right] \\ = \;& e^t(\tfrac{1}{4} + 0 + 1) \;=\; \tfrac{5}{4}e^t \quad \text{(making use of the fact that } \sin^2 t + \cos^t = 1\text{).} \end{aligned}$$

Thus

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \frac{\sqrt{5}}{2}e^{t/2}$$

and finally

$$L = \int_0^{\pi/2} \sqrt{\frac{5}{4}e^t}\, dt \quad \text{(worth 5 points)}$$

$$= \int_0^{\pi/2} \frac{\sqrt{5}}{2}e^{t/2}dt = \sqrt{5}\left[e^{t/2}\right]_0^{\pi/2} = \boxed{\sqrt{5}\left(e^{\pi/4} - 1\right)} = \boxed{2.668\cdots}.$$

(5 points) Find the equation of the line tangent to the curve at $t = 0$.

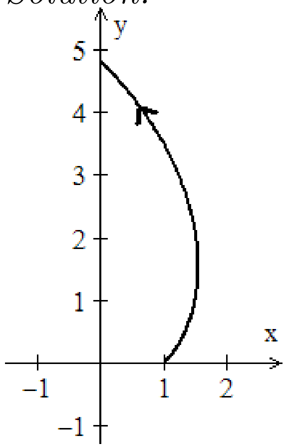
Solution. $x'(0) = \frac{1}{2}$, $y'(0) = 1$, and

$$\frac{dy}{dx} = \frac{y'(0)}{x'(0)} = \frac{1}{1/2} = 2.$$

The point we're considering is $(x(0), y(0)) = (1, 0)$, so we want the line through that point with slope 2. That is $y - 0 = 2(x - 1)$, or $\boxed{y = 2x - 2}$.

(4 points) Sketch the curve. Draw an arrow in the direction of increasing t.

Solution.



I'll be lenient in grading the graph: I'll give full credit for any arc that goes in a counterclockwise direction from the positive x-axis to the positive y-axis. But, properly speaking, it really ought to be increasing in radius, and the initial slope ought to be 2 (as seen in the previous problem).

Some students, by the time they got to this problem, forgot entirely that the interval was $0 \le t \le \pi/2$, and so they simply drew an arc that goes in counterclockwise direction, some distance around the origin. I gave 3 points for that.