Name: **Answer key**

Math 170 Final exam, 30 April 2010, 8 pages, 50 points, 120 minutes.

(5 points) Find the general solution of $x\dfrac{dy}{dx} + 2y = 1$.

Solve explicitly for y — that is, write your answer in the form y equals some function of just x and C. Simplify, to whatever extent you can.

Solution. Almost everyone got this one right. This can be solved by either of two methods.

Method 1. It's linear. Rewrite it in standard form as

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x}$$

Thus $P(x) = \frac{2}{x}$, so $\int P(x)dx = 2\ln x + C_1$, and we drop the C_1 for convenience. Then $I(x) = e^{\int P(X)dx} = e^{2\ln x} = x^2$. Multiply the standard form equation through by x^2, to obtain

$$x^2\frac{dy}{dx} + 2xy = x$$

$$\left(x^2y\right)' = x$$

$$x^2y = \int x dx$$

$$x^2y = \frac{1}{2}x^2 + C$$

$$\boxed{y = \tfrac{1}{2} + Cx^{-2}}.$$

Method 2. It has variables separable. Rewrite the equation as

$$x\frac{dy}{dx} = 1 - 2y$$

$$\frac{dy}{1-2y} = \frac{dx}{x}$$

$$\frac{dy}{2y-1} = -\frac{dx}{x}$$

$$\frac{dy}{y-\frac{1}{2}} = -2\frac{dx}{x}$$

Integrate both sides:

$$\ln\left|y - \frac{1}{2}\right| = -2\ln|x| + k$$

$$e^{\ln\left|y-\frac{1}{2}\right|} = e^k e^{-2\ln|x|}$$

$$\left|y - \frac{1}{2}\right| = e^k\left|x^{-2}\right|$$

$$(y - \frac{1}{2}) = \pm e^k x^{-2}$$

$$y - \frac{1}{2} = Cx^{-2}$$

$$\boxed{y = \tfrac{1}{2} + Cx^{-2}}.$$

(4 points) Find the area of the bounded region between the curves $y = x^2$ and $y = x^3$.

Solution. *Everyone* got this right, but I've already written the answer key and may as well include it here. First find the points where the two curves intersect: $x^2 = x^3$ yields $0 = x^3 - x^2 = x^2(x-1) = 0$, yielding, so the two intersections are at $x = 0$ and $x = 1$. Then the area is $\int_0^1 (x^2 - x^3)dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}} = \boxed{0.0833\ldots}$.

(5 points) Find the centroid of the bounded region between the curves $y = x^2$ and $y = x^3$.

Solution. Most got this right.

Chop the region up into thin slices, each approximated by a thin vertical rectangle. The rectangle corresponding to any particular value of x has

> center at $\left(x,\ \frac{x^2 + x^3}{2}\right)$

and it has

> height equal to $x^2 - x^3$, and width equal to dx, hence mass (area) equal to $(x^2 - x^3)dx$.

Thus the entire rectangle has

$$\text{moment} = (\text{mass})\,(\text{center}) = \left((x^3 - x^4)dx,\ \frac{x^4 - x^6}{2}dx\right).$$

Adding up all the rectangles, the entire region has moment equal to

$$\left(\int_0^1 (x^3 - x^4)dx,\ \int_0^1 \frac{x^4 - x^6}{2}dx\right) = \left(\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1,\ \left[\frac{x^5}{10} - \frac{x^7}{14}\right]_0^1\right)$$

$$= \left(\frac{1}{4} - \frac{1}{5},\ \frac{1}{10} - \frac{1}{14}\right) = \left(\frac{1}{20},\ \frac{1}{35}\right).$$

Stopping there was worth 4 points.

Finally, the centroid of the region is its moment divided by its mass, so we refer to the previous problem.

$$\left(\frac{1}{20}, \frac{1}{35}\right) / \frac{1}{12} = \left(\frac{12}{20}, \frac{12}{35}\right) = \boxed{\left(\frac{3}{5}, \frac{12}{35}\right)} = \boxed{(0.6,\ 0.3428\ldots)}.$$

(5 points) Find the surface area obtained by rotating the curve

$$y = \frac{3}{4}x^{2/3} - \frac{3}{8}x^{4/3} \qquad (0 \le x \le 1)$$

around the y-axis.

Solution. A few students made errors of one sort or another, but I was pleased to find that most students actually got this one right. Many of them did not show computation in evaluating their integral; I'm guessing they used a calculator to evaluate it for them.

In the following computation, I'm putting in some extra steps, in case the procedure isn't clear to someone.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2}x^{-1/3} - \frac{1}{2}x^{1/3} \\
\left(\frac{dy}{dx}\right)^2 &= \frac{1}{4}x^{-2/3} - \frac{1}{2} + \frac{1}{4}x^{2/3} &&= \left(\frac{1}{2}x^{-1/3} - \frac{1}{2}x^{1/3}\right)^2 \\
\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 &= \frac{1}{4}x^{-2/3} + \frac{1}{2} + \frac{1}{4}x^{2/3} &&= \left(\frac{1}{2}x^{-1/3} + \frac{1}{2}x^{1/3}\right)^2 \\
\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{\frac{1}{4}x^{-2/3} + \frac{1}{2} + \frac{1}{4}x^{2/3}} &&= \frac{1}{2}x^{-1/3} + \frac{1}{2}x^{1/3}
\end{aligned}$$

As we're rotating around the y-axis, we have $r = x$. (One student used $r = y$, which is entirely the wrong idea and gets the wrong answer.) Thus

$$A = \int_0^1 2\pi r\, ds = \int_0^1 2\pi x \frac{ds}{dx} dx = \int_0^1 2\pi x \sqrt{\frac{1}{4}x^{-2/3} + \frac{1}{2} + \frac{1}{4}x^{2/3}}\, dx$$

$$= \int_0^1 2\pi x \left[\frac{1}{2}x^{-1/3} + \frac{1}{2}x^{1/3}\right] dx = \pi \int_0^1 \left[x^{2/3} + x^{4/3}\right] dx$$

$$= \pi \left[\frac{3}{5}x^{5/3} + \frac{3}{7}x^{7/3}\right]_0^1 = \pi\left(\frac{3}{5} + \frac{3}{7}\right) = \boxed{\frac{36\pi}{35}} = \boxed{3.231\ldots}.$$

One student tried to reformulate everything as a function of y. I'm not sure why. It can be done, but it's a lot of extra work. Here's how to do it:

$$\begin{aligned} \tfrac{3}{4}x^{2/3} - \tfrac{3}{8}x^{4/3} &= y \\ x^{4/3} - 2x^{2/3} &= -\tfrac{8}{3}y \\ \left(x^{2/3} - 1\right)^2 &= 1 - \tfrac{8}{3}y \\ x &= \left[1 + \sqrt{1 - \tfrac{8}{3}y}\right]^{3/2} \end{aligned}$$

$$\tfrac{dx}{dy} = \tfrac{3}{2}\left[1 + \sqrt{1 - \tfrac{8}{3}y}\right]^{1/2} \cdot \tfrac{1}{2}\left(1 - \tfrac{8}{3}y\right)^{-1/2} \cdot \left(-\tfrac{8}{3}\right) = -2\sqrt{\tfrac{1+\sqrt{1-\frac{8}{3}y}}{1-\frac{8}{3}y}} = -2\sqrt{\tfrac{1}{1-\frac{8}{3}y} + \tfrac{1}{\sqrt{1-\frac{8}{3}y}}}$$

$$\left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2 = 1 + 4\left(\frac{1}{\sqrt{1 - \frac{8}{3}y}} + \frac{1}{1 - \frac{8}{3}y}\right) = \left(1 + \frac{2}{\sqrt{1 - \frac{8}{3}y}}\right)^2$$

And the endpoints of the curve are at $(0,0)$ and $(1, 3/8)$, so we get

$$A = \int_0^{3/8} 2\pi \left[1 + \sqrt{1 - \frac{8}{3}y}\right]^{3/2} \left(1 + \frac{2}{\sqrt{1 - \frac{8}{3}y}}\right) dy$$

and so on. I have not attempted to evaluate the integral, but it should end up with the same result, unless I've already made an arithmetic error.

(5 points) Find the area of one leaf of the four-leaved rose $r = \sin 2\theta$.

Hint: Some of these trigonometric identities may be helpful:

$$\cos 2x \;=\; 2\cos^2 x - 1 \;=\; \cos^2 x - \sin^2 x \;=\; 1 - 2\sin^2 x$$

Solution. First figure out that the curve crosses through the origin whenever $r = 0$. That is, whenever $\sin 2\theta = 0$. That's when $2\theta = n\pi$ for some integer n, or when $\theta = n\pi/2$. For instance, two of the consecutive crossings are at $\theta = 0$

and $\theta = \pi/2$, so the leaf in the first quadrant is $r = \sin 2\theta$ $(0 \le \theta \le \pi/2)$. Now compute

$$A = \int_0^{\pi/2} \frac{1}{2} r^2 d\theta = \int_0^{\pi/2} \frac{1}{2} \sin^2 2\theta \, d\theta$$

(4 points for getting that far correctly)

$$= \int_0^{\pi/2} \frac{1 - \cos 4\theta}{4} d\theta = \left[\frac{1}{4}\theta + \frac{1}{16}\sin 4\theta\right]_0^{\pi/2} = \boxed{\frac{\pi}{8}} = \boxed{0.3926\cdots}.$$

(5 points) For what three values of θ does the cardioid $r = 1 + \sin\theta$ $(0 \le \theta \le 2\pi)$ have a horizontal tangent line?

Solution. We're looking for places where $\dfrac{dy}{dx} = \dfrac{y'(\theta)}{x'(\theta)}$ becomes zero that is, where $y'(\theta) = 0$. From $y = r\sin\theta$, compute

$$y' = r'\sin\theta + r\cos\theta = \cos\theta\sin\theta + (1 + \sin\theta)\cos\theta = (\cos\theta)(1 + 2\sin\theta)$$

which vanishes when $\cos\theta = 0$ or when $\sin\theta = -1/2$. We get $\cos\theta = 0$ at the values $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, and we get $\sin\theta = -1/2$ when $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$. However, $\theta = \frac{3\pi}{2}$ is an extraneous solution — that's the point where $r = 0$, and the curve has *no* tangent line there. Thus the answer is $\boxed{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}}$.

Full credit for those three answers and no others. Deduct one point for each of those answers that is missing. Also deduct one point for each additional incorrect answer.

(12 points) In each of the following rows, check ($\surd$) either the "convergent" column or the "divergent" column.

series	conv	divg	explanation (not required)
$\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$		$\surd$	The integral test and the substitution $u = \ln x$ together yield $$\int_2^{\infty} \frac{dx}{x \ln x} = \int_{\ln 2}^{\infty} \frac{du}{u} = \lim_{R \to \infty} [\ln u]_{\ln 2}^{R} = \infty.$$
$\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$	$\surd$		because it is the telescoping series $\displaystyle\frac{1}{2}\sum_{n=2}^{\infty}\left[\frac{1}{n-1} - \frac{1}{n+1}\right]$, or because for n sufficiently large we have $n^2 - 1 > \frac{1}{2}n^2$, and therefore $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1} < 2\sum_{n=2}^{\infty} \frac{1}{n^2}$, which converges by the p-series test with $p = 2$.
$\displaystyle\sum_{n=1}^{\infty} \sqrt{n} \sin \frac{1}{n}$		$\surd$	by the limit comparison test, because when $n \to \infty$ the ratio between $\sin(1/n)$ and $1/n$ tends to 1, and $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges by the p-series test with $p = 1/2$.
$\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$	$\surd$		because it is absolutely convergent, because $\displaystyle\sum_{n=1}^{\infty} \frac{\lvert \sin(n) \rvert}{n^3}$ is convergent by comparison with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$, which is convergent by the p-series test with $p = 3$.

I gave no partial credit on this problem; there were four parts, each worth 3 points. (Actually, I was tempted to use a harsher scoring system: On true/false questions, even a person with no knowledge at all will get half of them right on the average, so it occurred to me that it might be more appropriate to deduct 6 points for each wrong answer – i.e., no credit at all if you get half of the problem wrong. But probably only a mathematician can appreciate that reasoning, so I left it at 3 points per question.)

(5 points) The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}x^n}{n} = x - \frac{3}{2}x^2 + \frac{9}{3}x^3 - \frac{27}{4}x^4 + \cdots$ has what interval of convergence?

Solution. We can apply the ratio test, with $a_n = \dfrac{(-3)^{n-1}x^n}{n}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = |3x|\frac{n}{n+1} \to |3x| = \begin{cases} > 1 & \text{if } |x| > 1/3, \\ < 1 & \text{if } |x| < 1/3. \end{cases}$$

We still need to consider what happens when $|x| = 1/3$.

At $x = -1/3$, the series is $-\frac{1}{3}(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots)$, so it's a constant times the harmonic series, and it is divergent.

At $x = 1/3$, the series is $\frac{1}{3}(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots)$, so it's a constant times the alternating harmonic series, and it is divergent.

Thus the interval of convergence is $\boxed{\left(-\dfrac{1}{3}, \dfrac{1}{3}\right]}$.

Partial credit: The answer has four ingredients: left parenthesis, $-\frac{1}{3}$, $\frac{1}{3}$, right bracket. Deduct one point for each ingredient that is incorrect.

(4 points) Write the power series for $\dfrac{x}{1+x^2}$. (*Hint*: How can you make this fit the formula for a geometric series?)

Solution. The geometric series is

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r} \quad \text{if } |r| < 1.$$

In particular, taking $a = x$ and $r = -x^2$ (worth 2 points), we get $\dfrac{x}{1+x^2}$ equal to

$$\boxed{x - x^3 + x^5 - x^7 + x^9 - \cdots} \text{ or } \boxed{\sum_{k=0}^{\infty} (-1)^k x^{2k+1}} \text{ or } \boxed{\sum_{k=1}^{\infty} (-1)^{k-1} x^{2k-1}}.$$

Common errors: Deduct one point for not having the signs alternating, or having them all flipped wrong. Deduct one point for omitting one or two terms at the beginning of the series, or for omitting the subscript altogether on the $\sum$ symbol.

Three students had perfect scores. The average score for the class was 43 points out of 50; that's 86 percent.

The class's average semester score was 86.51%, which is a grade of B.