Hyperplane arrangements, flag complexes and monoid cohomology

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Happy Birthday Stuart Conference

Outline

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- Then we try to put it altogether and state our main results.

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- $|K|$ is the union of the simplices spanned by sets of coordinate vectors corresponding to an element of $\mathcal{F}$.

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- Then $\Delta(P)$ is the barycentric subdivision of K.

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- The nerve of an open cover is fundamental to Čech cohomology.

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- The modern way to formulate his result is via Leray numbers.

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- E.g., $\mathrm{vd}_{\Bbbk}(S^1 \times [0,1]^2) = 2 = \mathrm{vd}_{\Bbbk}(S^1)$.
- $\mathrm{vd}_{\Bbbk}(K)$ is a homotopy invariant.

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- $\mathrm{Ler}_{\Bbbk}(K) = 0$ iff K is a simplex.

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- The clique complex $\mathrm{Cliq}(G)$ is the flag complex with vertex set V and simplices the cliques of G.

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Cliq(G) is 1-representable iff G is chordal and $\overline{G}$ is a comparability graph (Lekkerkerker, Boland).

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- You factor the polynomial ring on the vertices by the ideal generated by non-faces.
- The Leray number $\operatorname{Ler}_{\Bbbk}(K)$ turns out to be the Castelnuovo-Mumford regularity of the Stanley-Reisner ring.

Left regular bands (LRBs)

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- All LRBs are assumed finite with identity.

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- Markov chains on these objects can be analyzed via LRB representation theory.

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Others:

Björner, Athanasiadis-Diaconis, Chung-Graham, . . .

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  - long-term behavior: favorite books move to the front

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This construction generalizes to matroids and interval greedoids.

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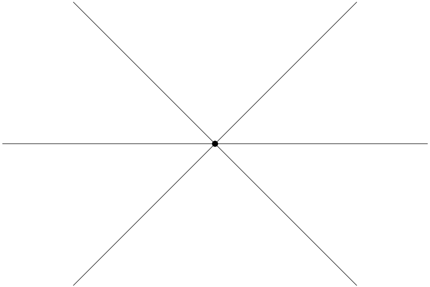

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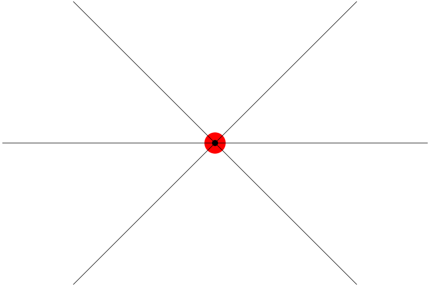

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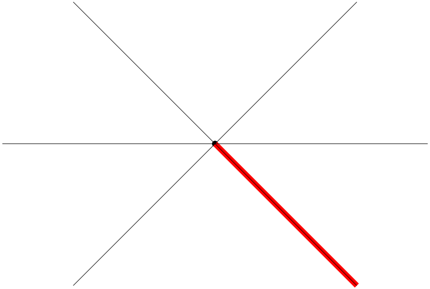

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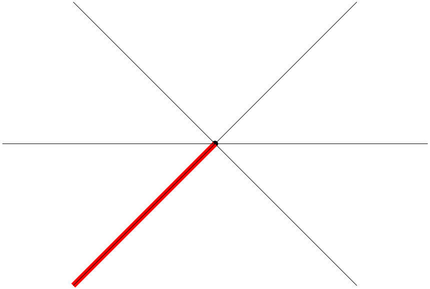

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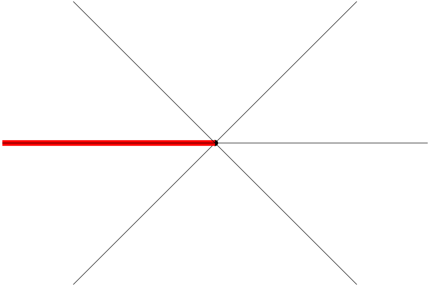

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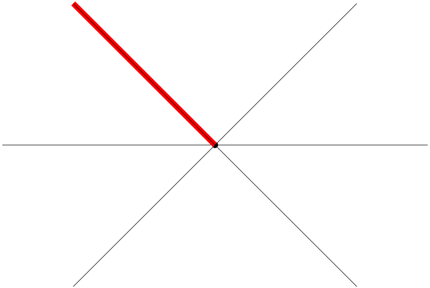

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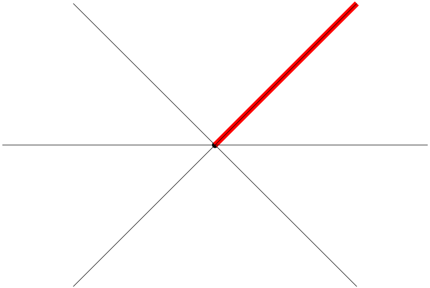

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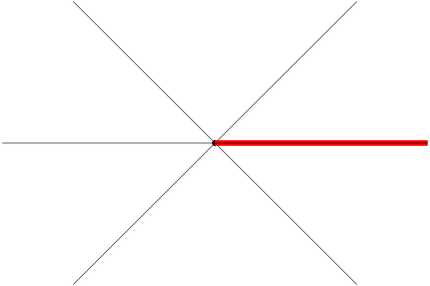

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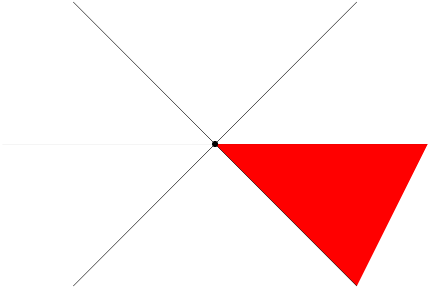

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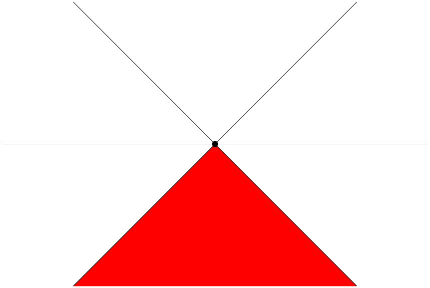

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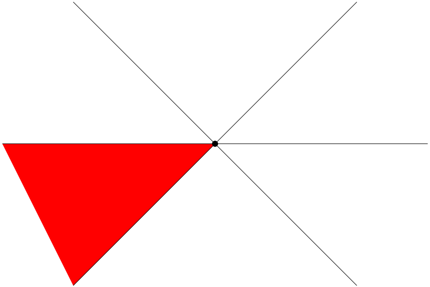

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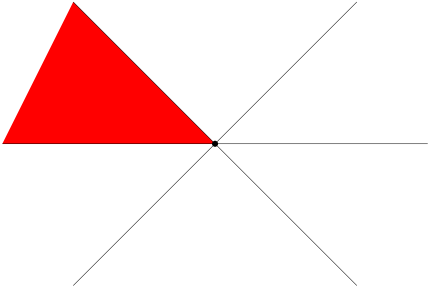

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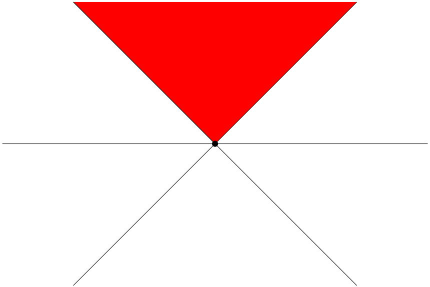

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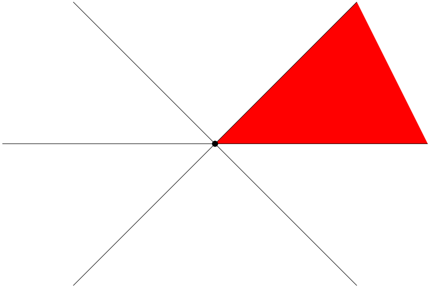

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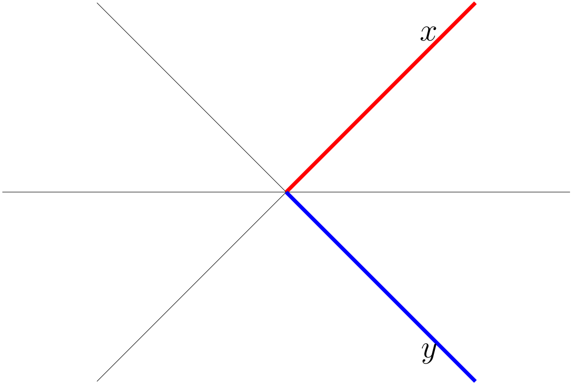
$$xy := \begin{cases}\text{the face first encountered after a small} \\ \text{movement along a line from } x \text{ toward } y\end{cases}$$



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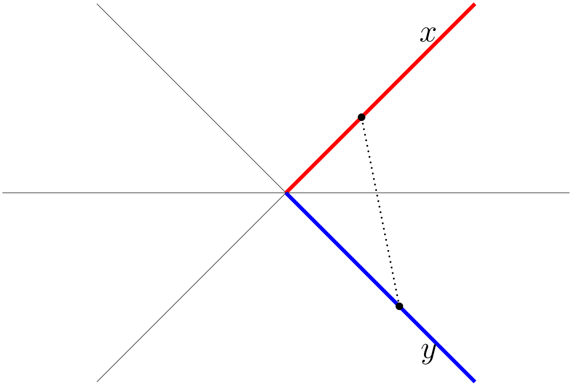

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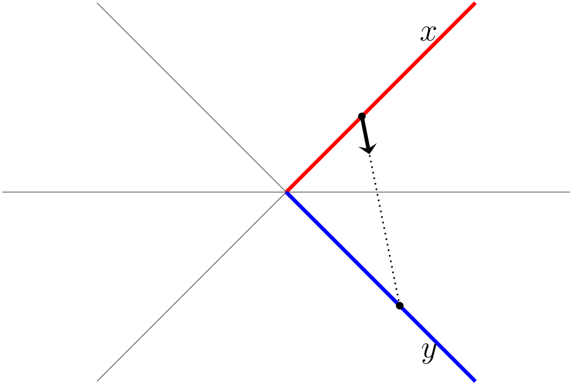

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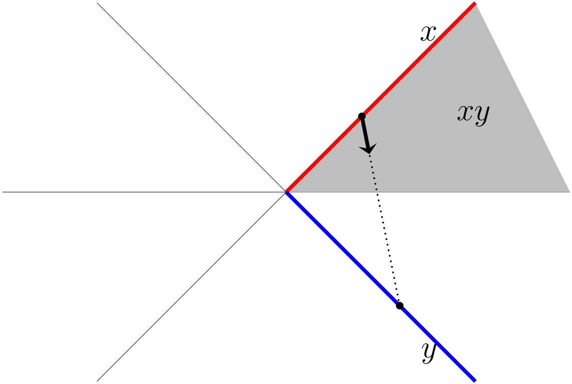

Solomon's descent algebra

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- For instance, in type A the algebra $\Sigma(W)$ maps onto the character ring with nilpotent kernel.

Free partially commutative LRBs

- The free partially commutative LRB $B(G)$ on a graph $G = (V, E)$ is the LRB with presentation:

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- These are LRB-analogues of free partially commutative monoids and groups.

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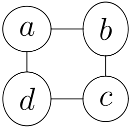

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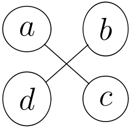

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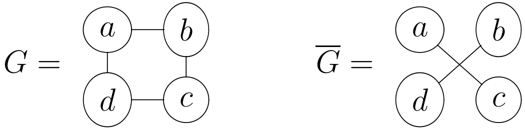

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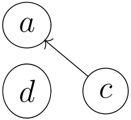

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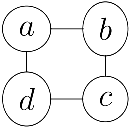
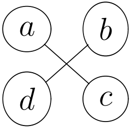
$G =$ 

$\overline{G} =$ 

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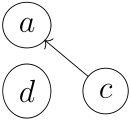
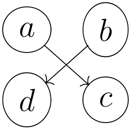


In $B(G)$: $cad = cda = dca$ (c comes before a since $c \to a$)

Random walk on $B(G)$

States: acyclic orientations of the complement $\overline{G}$



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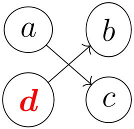

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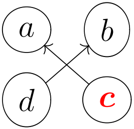

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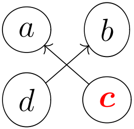

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Athanasiadis-Diaconis (2010): studied this chain using a different LRB (graphical arrangement of $\overline{G}$)

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- Equivalently, $H^n(M;A) = \mathrm{Ext}^n_{\mathbb{Z}M}(\mathbb{Z},A)$.

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- We were able to reduce this question to computing $\mathrm{cd}_{\Bbbk}$ of LRBs.

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Let $\mathscr{H}$ be an essential hyperplane arrangement in $\mathbb{R}^d$ with corresponding face monoid $\mathcal{F}(\mathscr{H})$. Then $\mathrm{cd}_{\Bbbk}(\mathcal{F}(\mathscr{H})) = d$.

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Corollary

Let $\mathscr{H}$ and $\mathscr{H}'$ be essential hyperplane arrangements in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $M = \mathcal{F}(\mathscr{H}) \times \mathcal{F}(\mathscr{H}')^{op}$. Then $\mathrm{cd}(M) = n$ and $\mathrm{cd}(M^{op}) = m$.

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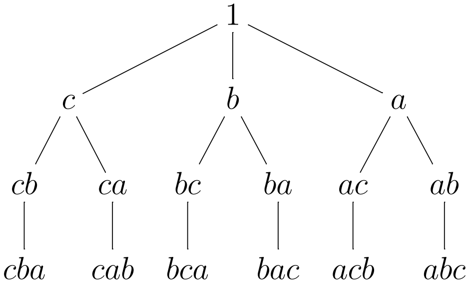
Example: $F(\{a, b, c\})$



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- Thus $\Delta(\mathcal{F}(\mathscr{H}))$ is the barycentric subdivision of $\mathcal{Z}(\mathscr{H})$.

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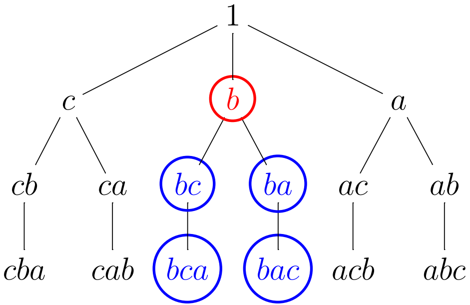
Example: $F(\{a,b,c\})_{<b}$ is given by



$$F(\{a,b,c\})_{<b} = \{bc, ba, bca, bac\}$$

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- $\Delta(\mathcal{F}(\mathscr{H})_{<a})$ is in general a sphere of dimension $< d$.

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- Moreover, each induced subcomplex comes up in this way.

The main theorem

Recall: $\mathrm{vd}_{\Bbbk}(K) = \min\{d \mid \forall n \geq d, \widetilde{H}^n(K, \Bbbk) = 0\}$.

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Semi-free actions

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Theorem (MSS)

Suppose that $B \curvearrowright K$ is a semi-free action on a contractible simplicial complex. Then the augmented chain complex of K is a projective resolution of $\mathbb{k}$ over $\mathbb{k}B$.

The action on $\Delta(B)$

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- The last isomorphism uses that $B_{\leq a}$ is a cone on $B_{<a}$, hence contractible, and the long exact sequence in relative cohomology.

The end

Thank you for your attention!