Fixed points for groups and monoids

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Ramat Gan, 11th June 2013

Part of the results in this talk were obtained in collaboration with:

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Fixed points

- If M is a finitely generated monoid and $\varphi \in \operatorname{End} M$, then

$$\operatorname{Fix} \varphi = \{x \in M \mid x\varphi = x\}$$

  is the submonoid of fixed points
- $\operatorname{Per} \varphi = \cup_{n \geq 1} \operatorname{Fix} \varphi^n$ is the submonoid of periodic points
- If M is a group, both $\operatorname{Fix} \varphi$ and $\operatorname{Per} \varphi$ are subgroups

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- If M is a group, both $\text{Fix}\,\varphi$ and $\text{Per}\,\varphi$ are subgroups
- If d is a metric on M inducing the discrete topology, we are also interested in the study of $\text{Fix}\,\Phi$ if there exists a continuous extension Φ of $\varphi \in \text{End}\, M$ to the completion $\widehat{M}$
- The fixed points in the topological closure $\overline{\text{Fix}\,\varphi}$ are said to be singular, the other ones are regular

Landmarks

- Gersten 1984: $\mathrm{Fix}\,\varphi$ is finitely generated when G is a free group and $\varphi \in \mathrm{Aut}\,G$
- Goldstein and Turner 1986: $\mathrm{Fix}\,\varphi$ is finitely generated when G is a free group and $\varphi \in \mathrm{End}\,G$

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- Cooper 1987: $\mathrm{Reg}\,\Phi$ is a finite union of $(\mathrm{Fix}\,\varphi)$-orbits when G is a free group under the prefix metric and $\varphi \in \mathrm{Aut}\,G$
- Paulin 1989: $\mathrm{Fix}\,\varphi$ is finitely generated when G is a hyperbolic group and $\varphi \in \mathrm{Aut}\,G$

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- Paulin 1989: $\mathrm{Fix}\,\varphi$ is finitely generated when G is a hyperbolic group and $\varphi \in \mathrm{Aut}\,G$
- Bestvina and Handel 1992: $\mathrm{rk}\,\mathrm{Fix}\,\varphi \leq n$ when G is a free group of rank n and $\varphi \in \mathrm{Aut}\,G$
- Gaboriau, Jaeger, Levitt and Lustig 1998: if G is a free group under the prefix metric and $\varphi \in \mathrm{Aut}\,G$, then every $\alpha \in \mathrm{Reg}\,\Phi$ is either an attractor or a repeller

Previous work

- Cassaigne and PVS (*Ann. Inst. Fourier* 2009): dynamics of $\mathrm{Reg}\,\Phi$ when M is a monoid defined by a special confluent rewriting system, d is the prefix metric and φ is either a prefix-convergent or an expanding endomorphism

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- PVS (*Monatshefte Math.* 2010): $\mathrm{Fix}\,\varphi$ is rational if M is a monoid defined by a special confluent rewriting system and the endomorphism φ is either boundary-injective or has bounded length decrease; finiteness theorems for $\mathrm{Reg}\,\Phi$

Inverse transducers

- A finite A-transducer is a finite A-automaton with an output function for edges
- Edges are labelled $p \xrightarrow{a|u} q$ with $a \in A$ and $u \in A^*$
- An A-transducer $\mathcal{T}$ induces a partial mapping $\eta_{\mathcal{T}} : \widetilde{A}^* \to \widetilde{A}^*$ through the labels of successful paths

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- An A-transducer $\mathcal{T}$ induces a partial mapping $\eta_{\mathcal{T}} : \widetilde{A}^* \to \widetilde{A}^*$ through the labels of successful paths
- If $\widetilde{A} = A \cup A^{-1}$ and the $\widetilde{A}$-transducer $\mathcal{T}$ is deterministic, complete and satisfies

$$p \xrightarrow{a|u} q \text{ if and only if } q \xrightarrow{a^{-1}|u^{-1}} p,$$

  it is said to be inverse
- if $\mathcal{T}$ is inverse, $\eta_{\mathcal{T}}$ induces a partial mapping $\overline{\eta}_{\mathcal{T}} : F_A \to F_A$ (a transduction of the free group F_A)

A finiteness theorem

Theorem (PVS 2012)

Let ψ be a transduction of F_A and let $z \in F_A$. Then

$$L_z = \{g \in F_A \mid g\psi = gz\}$$

is rational

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- We define an infinite inverse $\widetilde{A}$-automaton $\mathcal{A}$ with vertices $(g^{-1}(g\psi), q_0 g)$
- We define the outward edges of $\mathcal{A}$

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- We define an infinite inverse $\widetilde{A}$-automaton $\mathcal{A}$ with vertices $(g^{-1}(g\psi), q_0 g)$
- We define the outward edges of $\mathcal{A}$
- Then we use them to show that there is a finite subautomaton of $\mathcal{A}$ recognizing the reduced forms of L_z

Virtually free groups

- A group is virtually free if it has a free subgroup of finite index
- It is straightforward to derive from the preceding theorem an alternative proof for:

Theorem (Sykiotis 2002)

Let φ be an endomorphism of a finitely generated virtually free group. Then $\mathrm{Fix}\,\varphi$ is finitely generated.

A new model for the boundary

- Virtually free groups are hyperbolic and have thus a well-known established boundary
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- By using a result of Gilman, Hermiller, Holt and Rees, we can choose A so that:
  - M_A is the set of irreducibles of a nice finite rewriting system
  - M_A under the prefix metric can be completed by adding all the infinite words with prefixes in M_A
  - This completion, under the prefix metric, is homeomorphic to the hyperbolic completion $\widehat{G}$ of G

Uniformly continuous endomorphisms

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- We can prove that, for virtually free groups, they are precisely the virtually injective endomorphisms
- Moreover, they satisfy the bounded cancellation property

Finitely many orbits

- $\mathrm{Fix}\,\varphi$ acts naturally on the left of $\mathrm{Fix}\,\Phi$
- By restriction, $\mathrm{Fix}\,\varphi$ acts also on the left of $\mathrm{Sing}\,\Phi$ and $\mathrm{Reg}\,\Phi$
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Theorem (PVS 2012)

Let φ be a virtually injective endomorphism of a finitely generated virtually free group G. Then $\operatorname{Reg}\Phi$ has finitely many $(\operatorname{Fix}\varphi)$-orbits.

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Corollary (PVS 2012)

Let φ be a virtually injective endomorphism of a finitely generated virtually free group G with $\mathrm{Fix}\,\varphi$ finite. Then $\mathrm{Fix}\,\Phi$ is also finite.

Classification of the regular fixed points

The following theorem generalizes the theorem proved by Gaboriau, Jaeger, Levitt and Lustig for free group automorphisms:

Theorem (PVS 2012)

Let φ be an automorphism of a finitely generated virtually free group. Then $\text{Reg}\,\Phi$ contains only exponentially stable attractors and exponentially stable repellers.

Graph groups

- Let $\Gamma = (V, E)$ be a finite simple graph
- The graph group $G(\Gamma)$ is presented by

$$\langle V \mid ab = ba, \ (a \text{ --- } b) \in E \}$$

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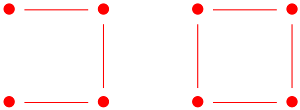
- Graph groups are also known as right-angled Artin groups
- We say that a simple graph is a transitive forest if it has no full subgraphs of one of the following forms



Endomorphism fixed points

Theorem (Rodaro, PVS and Sykiotis 2012)

Let $\Gamma = (V, E)$ be a finite simple graph. Then the following conditions are equivalent:

(i) $\mathrm{Fix}\,\varphi$ is finitely generated for every $\varphi \in \mathrm{End}\, G(\Gamma)$;

(ii) $\mathrm{Per}\,\varphi$ is finitely generated for every $\varphi \in \mathrm{End}\, G(\Gamma)$;

(iii) Γ is a disjoint union of complete graphs;

(iv) $G(\Gamma)$ is a free product of finitely many free abelian groups of finite rank.

Automorphism fixed points

Theorem (Rodaro, PVS and Sykiotis 2012)

Let $\Gamma = (V, E)$ be a finite transitive forest. Then the following conditions are equivalent:

(i) $\text{Fix}\,\varphi$ is finitely generated for every $\varphi \in \text{Aut}\, G(\Gamma)$;

(ii) $\text{Per}\,\varphi$ is finitely generated for every $\varphi \in \text{Aut}\, G(\Gamma)$;

(iii) Γ is a disjoint union of complete graphs;

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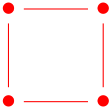
What if the graph is not a transitive forest?

In this case, it can go either way:

- If Γ is the square



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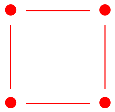

Fix φ and Per φ are always finitely generated

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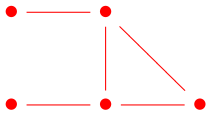

both Fix φ and Per φ may be non finitely generated

Techniques used

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- If Γ is not a disjoint union of complete graphs, we explicitly construct endomorphisms/automorphisms with non finitely generated fixed (periodic) point subgroups
- If Γ is a disjoint union of complete graphs, we use theorems of Sykiotis on the Kurosh rank of free products of finitely generated nilpotent and finite groups
- Sykiotis' theorems (2005 and 2007) have generalized the aforementioned Bestvina and Handel's rank theorems

Trace monoids

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- Trace monoids are the monoid version of graph groups
- Foata normal form: products of blocks $w_1 \dots w_k$ where
  - all letters in a block are different and commute
  - if a occurs in w_{i+1} then in w_i occurs either a or some letter not commuting with a

Fixed and periodic points

Theorem (Rodaro and PVS 2012)

Let $\Gamma = (V, E)$ be a finite simple graph and let $\varphi \in \operatorname{End} M(\Gamma)$. Then:

(i) $\operatorname{Fix} \varphi$ is finitely generated and effectively computable;

(ii) $\operatorname{Per} \varphi$ is finitely generated and effectively computable.

Techniques used: combinatorics on traces

Real traces

- A poset is d-complete if every directed set admits a join
- The prefix order is a partial order on $M(\Gamma)$
- The ideal completion $\mathbb{R}(\Gamma)$ of $M(\Gamma)$ is the set of real traces

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- A poset is d-complete if every directed set admits a join
- The prefix order is a partial order on $M(\Gamma)$
- The ideal completion $\mathbb{R}(\Gamma)$ of $M(\Gamma)$ is the set of real traces
- $\mathbb{R}(\Gamma)$ is d-complete and every order-preserving mapping φ of $M(\Gamma)$ admits a unique (Scott) continuous extension Φ to $\mathbb{R}(\Gamma)$ (Φ preserves directed sets and their joins)

Finiteness conditions

- $\mathbb{R}(\Gamma)$ is best described as the set of all finite and infinite traces arising from Γ
- The Foata normal form can be generalized to produce a normal form $w_1 w_2 \ldots$ for infinite traces

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- $\mathbb{R}(\Gamma)$ is best described as the set of all finite and infinite traces arising from Γ
- The Foata normal form can be generalized to produce a normal form $w_1 w_2 \ldots$ for infinite traces
- We say that $Y \subseteq \mathbb{R}(\Gamma)$ is rational if Y can be obtained from finite subsets of $\mathbb{R}(\Gamma)$ by applying finitely many times the operators union, product, star and mixed product (finite by infinite)
- Two infinite traces are suffix-equivalent if they share an (infinite) suffix

A finiteness theorem

Theorem (Rodaro and PVS 2012)

Let $\Gamma = (V, E)$ be a finite transitive forest. Then the following conditions are equivalent:

(i) for every $\varphi \in \operatorname{End} M(A, I)$, $\operatorname{Reg} \Phi$ is rational;

(ii) for every $\varphi \in \operatorname{End} M(A, I)$, $\operatorname{Reg} \Phi$ has only finitely many suffix-equivalence classes;

(iii) Γ is a disjoint union of complete graphs;

(iv) $M(\Gamma)$ is a free product of finitely many free commutative monoids of finite rank.

Techniques used: order theory and combinatorics on traces

Inverse monoids

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- Inverse monoids can be viewed as monoids of partial injective transformations closed under inversion
- The free inverse monoid on A (denoted as FI_A) admits as normal forms the set of all finite birooted A-labelled trees (Munn)
- $\theta : \widetilde{A}^* \to FI_A$ denotes the canonical homomorphism

Chomsky's hierarchy

For languages:

$$\begin{array}{lcl} \text{rational} & \subset & \text{context-free} \\ & \subset & \text{context-sensitive} \\ & \subset & \text{recursively enumerable} \end{array}$$

Chomsky's hierarchy

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For subsets of FI_A:

X is $\mathcal{C}$ if $X = L\theta$ for some L in $\mathcal{C}$

Periodic points

Theorem (Rodaro and PVS 2012)

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Periodic points

Theorem (Rodaro and PVS 2012)

Let $\varphi \in \text{End}\, FI_A$. Then $\text{Per}\,\varphi$ is finitely generated.

- Every $\varphi \in \text{End}\, FI_A$ induces some $\varphi' \in \text{End}\, F_A$
- If φ' in injective, it admits a (unique) continuous extension $\widehat{\varphi'} : \widehat{F_A} \to \widehat{F_A}$

A huge collapse

Theorem (Rodaro and PVS 2012)

Let $\varphi \in \operatorname{End} FI_A$ be such that φ' is injective and $\operatorname{Fix} \widehat{\varphi'} = 1$. Then the following conditions are equivalent:

(i) $\operatorname{Fix} \varphi$ is context-free;

(ii) $\operatorname{Fix} \varphi$ is rational;

(iii) $\operatorname{Fix} \varphi$ is finitely generated;

(iv) $\operatorname{Fix} \varphi$ is finite;

(v) $\operatorname{Per} \varphi$ is finite;

(vi) $\operatorname{Per} \varphi \subseteq E(FI_A)$.

Fixed points

Theorem (Rodaro and PVS 2012)

Let $\varphi \in \mathrm{End}\, FI_A$ permute $\widetilde{A}$ without fixing any letter. Then $\mathrm{Fix}\, \varphi$ is not context-free.

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Theorem (Rodaro and PVS 2012)

Let $\varphi \in \mathrm{End}\, FI_A$. Then $\mathrm{Fix}\, \varphi$ is context-sensitive.

Techniques used: combinatorics on trees, combinatorial group theory, language theory