The power of $\mathscr{R}$-trivial monoids

Anne Schilling

Department of Mathematics, UC Davis

based on

- Arvind Ayyer, Steve Klee, Anne Schilling, arXiv:1205.7074
- Arvind Ayyer, Anne Schilling, Ben Steinberg, Nicolas M. Thiéry, arXiv.1305.1697
- Arvind Ayyer, Anne Schilling, Ben Steinberg, Nicolas M. Thiéry, in preparation

Bar Ilan University, Israel, June 13, 2013

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- **Directed Nonabelian Sandpile Models:**
  Grain toppling on arborescences:
  - Nice stationary distributions and wreath product interpretation.
  - Integer eigenvalues and nice multiplicities!

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- **Representation Theory of Monoids:**
  - Use the representation theory of $\mathscr{R}$-trivial monoids.
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Also known as the Chip-firing game.

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- Set of leaves L: vertices with in-degree zero.

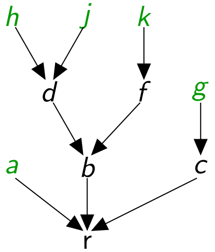

Figure: An arborescence with leaves at a, g, h, j, k.

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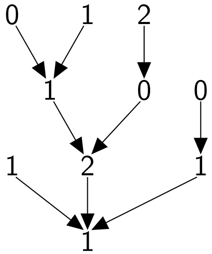

Figure: A configuration with all thresholds 2.

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- In both, sand grains enter at the leaves, ...
- ..., topple along the vertices, ...
- ..., and exit at the root.
- Unlike in the abelian sandpile model, sand grains only enter at leaves.
- The operators defining the entrance of sand grains are the same in both models.

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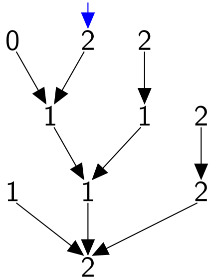
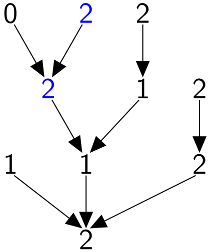
Follow the path $\ell^{\downarrow}$ from ℓ to the root r

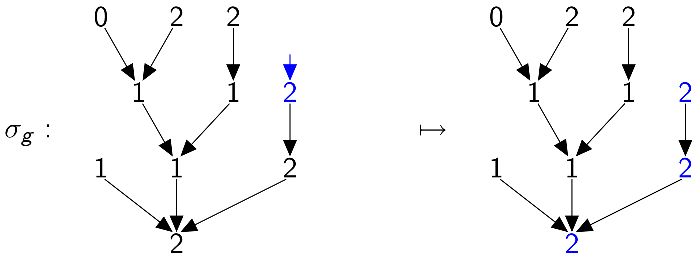
- Add a grain to the first vertex along the way that has not yet reached its threshold, if such a vertex exists.
- If no such vertex exists, then the grain is interpreted to have left the tree at the root and $\sigma_\ell(t) = t$.

σ_j :



$\mapsto$




σ_g :
0 2 2
1 1 2
1 1 2
2
↦
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2

Topple operators

Definition (Trickle-down sandpile model)

$$\theta_v : \Omega(\mathcal{T}) \to \Omega(\mathcal{T})$$

θ_v moves one grain from $v \in V$ to the first available site along $v^{\downarrow}$. If no such site exists, the grain exits the system.

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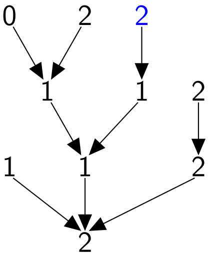
Remark

If $t_v = 0$ (no grain at site v), then $\theta_v(t) = \tau_v(t) = t$.

Toppling in the Trickle-down sandpile model

θ_k :



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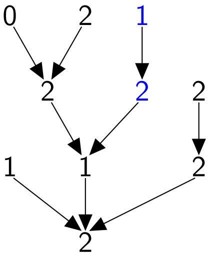

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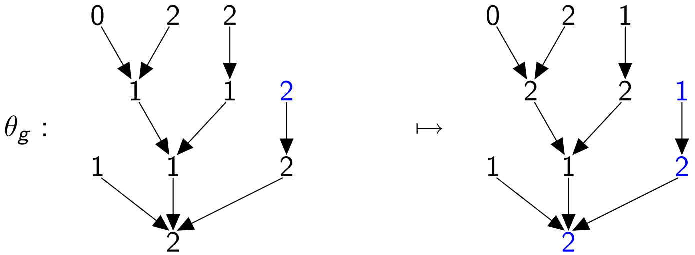

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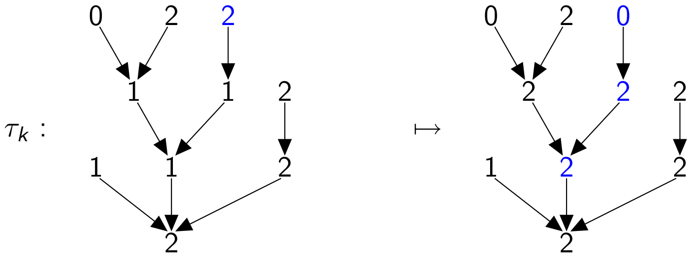

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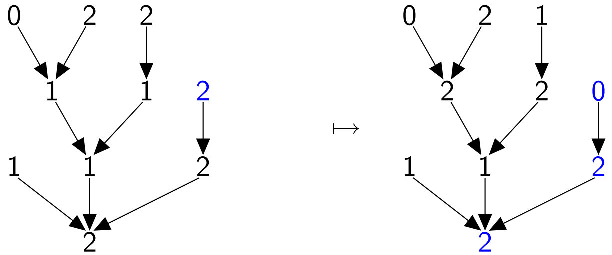
τ_g :



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  x_v: probability of choosing the topple operator θ_v (resp. τ_v)
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  We assume that
  1. $$0 < x_v, y_\ell \leq 1$$
  2. $$\sum_{v \in V} x_v + \sum_{\ell \in L} y_\ell = 1$$

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- Recursive definition: Both models can be defined recursively by successively removing leaves.
- Sources on all vertices: Allow source operators at all vertices, not just leaves!

Ergodicity

Proposition (ASST 2013)

G_θ: directed graph with

- *vertex set $\Omega(\mathcal{T})$*
- *edges given by σ_ℓ for $\ell \in L$ and θ_v for $v \in V$.*

Then G_θ is strongly connected and hence the Trickle-down sandpile model is ergodic.

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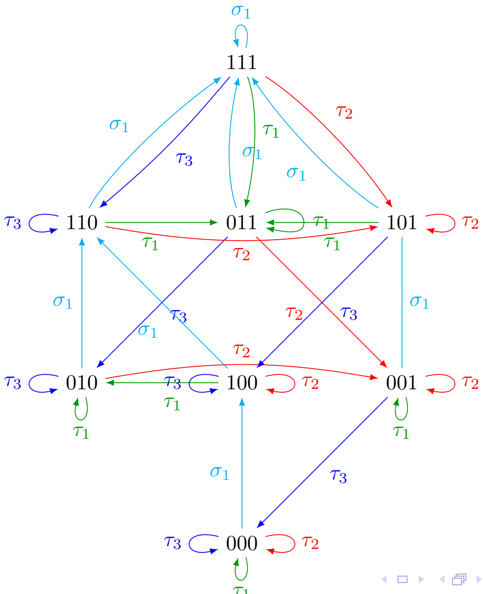
Then G_τ *is strongly connected and hence the Landslide sandpile model is ergodic.*

Markov chains on a line with thresholds 1



Trickle-down sandpile model: Stationary distribution

- $L_v := \{\ell \in L \mid v \text{ is a vertex of } \ell^{\downarrow}\}$
- $Y_v := \sum_{\ell \in L_v} y_\ell$
- For $0 \leq h \leq T_v$

$$\rho_v(h) := \frac{Y_v^h x_v^{T_v - h}}{\sum_{i=0}^{T_v} Y_v^i x_v^{T_v - i}}$$

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Theorem (ASST 2013)

The stationary distribution of the Trickle-down sandpile Markov chain defined on G_θ is given by the product measure

$$\mathbb{P}(t) = \prod_{v \in V} \rho_v(t_v).$$

Landslide sandpile model: Stationary distribution

$$\mu_v(h) := \begin{cases} \dfrac{Y_v^h x_v}{(Y_v + x_v)^{h+1}} & \text{if } h < T_v \\ \\ \dfrac{Y_v^{T_v}}{(Y_v + x_v)^{T_v}} & \text{if } h = T_v \end{cases}$$

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Theorem (AST 2013)

Let $T_v = 1$ for all $v \in V$, $v \neq \mathrm{r}$ and $T_{\mathrm{r}} = m$ for some positive integer m. Then the stationary distribution of the Landslide sandpile model defined on G_τ is given by the product measure

$$\mathbb{P}(t) = \prod_{v \in V} \mu_v(t_v).$$

Landslide sandpile model: Spectrum

For subsets $S \subseteq V$ and $\ell^{\downarrow}$ the set of vertices on path from ℓ to r:

$$y_S = \sum_{\ell \in L, \ell^{\downarrow} \subseteq S} y_\ell \quad \text{and} \quad x_S = \sum_{v \in S} x_v .$$

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The characteristic polynomial of M_τ is given by

$$\det(M_\tau - \lambda \mathbb{1}) = \prod_{S \subseteq V} (\lambda - (y_S + x_S))^{T_{S^c}},$$

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Eigenvalues: $y_S + x_S$

Multiplicities: T_{S^c}

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Mixing time is at most $\frac{2(n+c-1)}{p}$.

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Example

$w = 231231 \in \mathfrak{R}$ for S_4. Then $\partial_1(w) = 123121$ since $123123 = 121323 = 212323$ is not reduced!

Methods from the representation theory of monoids

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E.g.:

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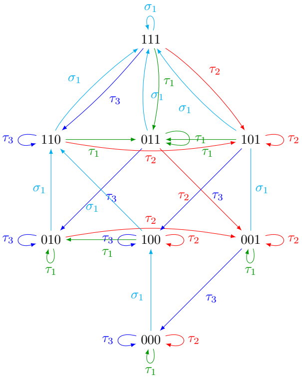
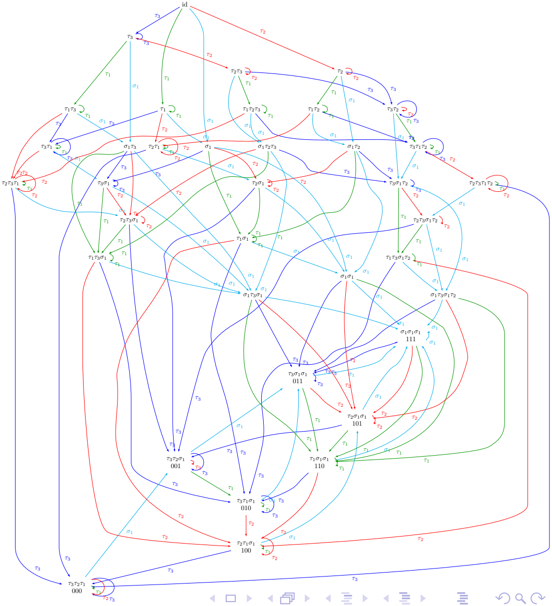
Alternatively from transition matrix $\overline{M}$ of Markov chain:

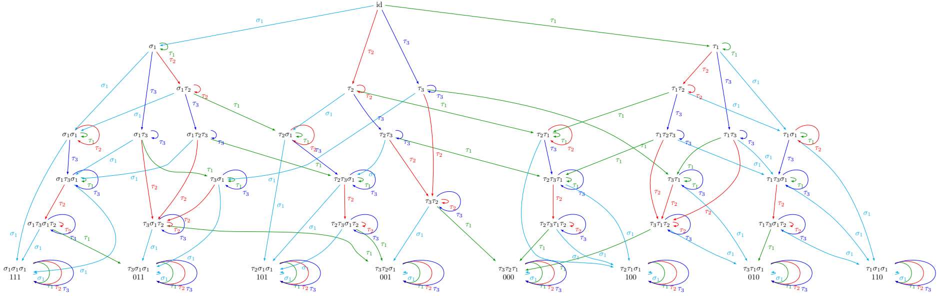
$$m_i = \overline{M}_{x_i=1; x_1=\cdots=x_{i-1}=x_{i+1}=\cdots=x_n=0}.$$

The left Cayley graph for the 1D sandpile model





The right Cayley graph for the 1D sandpile model



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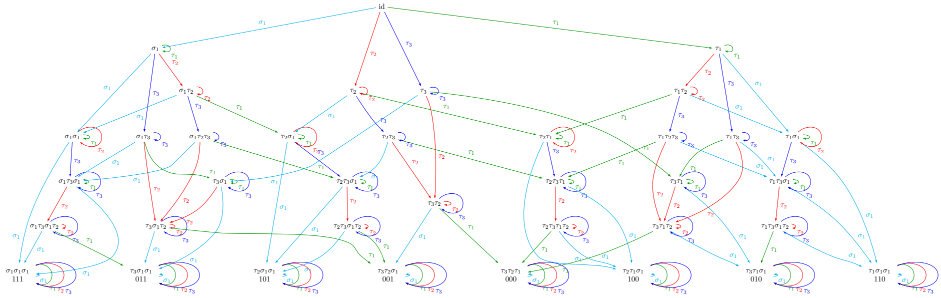

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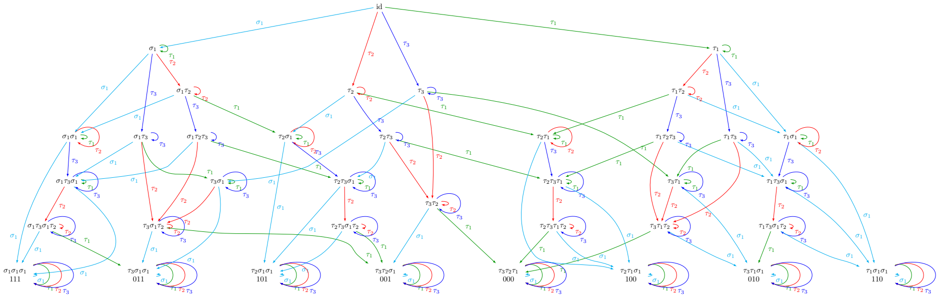

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- Not too deep $\implies$ bound on the rates of convergence

Definitions: Green relations

- Left and right preorders on $\mathcal{M}$:

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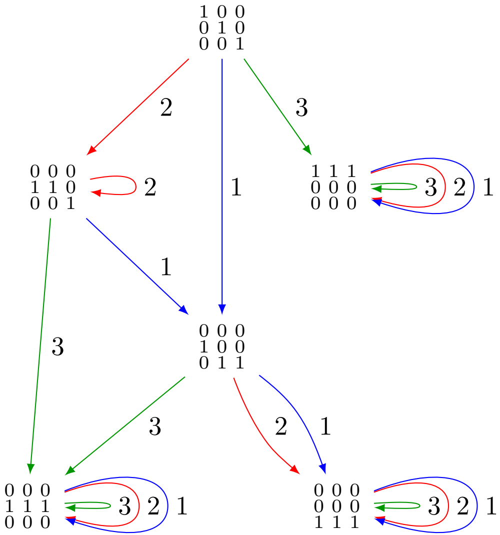
$$x\mathscr{L}y \quad \text{if} \quad \mathcal{M}y = \mathcal{M}x$$

Definition

$\mathcal{M}$ is $\mathscr{R}$-trivial ($\mathscr{L}$-trivial) if all $\mathscr{R}$-classes ($\mathscr{L}$-classes) are singletons. Equivalently, if the preorders are partial orders.

$\mathscr{R}$-trivial monoid for promotion example



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Method

- Show that $\mathcal{M}$ is $\mathscr{R}$-trivial
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- Simple modules are of dimension 1

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- Recover the composition factors using the character table

Markov chains and Representation Theory

The idea of decomposing the configuration space is not new!

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Wreath product formulation of sandpile models

Choose a leaf ℓ, and decompose the state space:

$$\Omega(\mathcal{T}) = \{0, \ldots, T_\ell\} \times \Omega(\nabla_\ell(T)),$$

where $\nabla_\ell \mathcal{T}$ is $\mathcal{T}$ without the leaf ℓ.

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Recursive definition (for Landslide sandpile model):

$$\sigma_\ell(t_\ell, t) = \begin{cases} (t_\ell + 1, t) & \text{if } t_\ell < T_\ell \\ (T_\ell, \sigma_{\mathbf{s}(\ell)} t) & \text{if } t_\ell = T_\ell \end{cases}$$

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Wreath product formulation of sandpile models (cont.)

Wreath product

$$M(\mathcal{T}) \subseteq (M(T_\ell), [T_\ell]) \wr (M(\nabla_\ell \mathcal{T}), \Omega(\nabla_\ell \mathcal{T}))$$

where

- $M(m) = \langle \alpha_m, \overline{0} \rangle$ with $\overline{0}$ the constant map 0 and

$$\alpha_m(h) = \begin{cases} h+1 & \text{if } h < m \\ m & \text{if } h = m \end{cases}$$

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$\Rightarrow$ useful for proof of $\mathscr{R}$-triviality

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$\mathcal{M}$ semigroup, X set of idempotent generators

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Alternative description:

X is a set of generators such that for each generator x, $x^{k+1} = x^k$ for some k

Require that there exists a total order $<_X$ on X such that

- x and y commute or
- x is idempotent and $xyx = xy$

whenever $x <_X y$.

Future work

$\mathscr{R}$-trivial machinery:

- Multitude of models fits this setting!

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Mixing time for linear extensions:

- Recall that the uniform promotion graph led to the uniform distribution on linear extensions
- Counting linear extensions is an important problem in practice.
- Can we get better bounds on cover times? Or mixing times? (since Markov chains are irreversible)
- Explicit conjecture for second largest eigenvalue for random-to-random shuffling on posets

Happy birthday Stuart !