Boolean representations of simplicial complexes

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Ramat Gan, 11th June 2013

The results presented in this talk are joint work with Pedro V. Silva (Porto):

Matrices

Simplicial complexes

Matroids

Graphs

Lattices

Partial euclidean geometries

Configurations

Abstract simplicial complexes

- Let V be a finite set and let $H \subseteq 2^V$
- (V, H) is a simplicial complex (or hereditary collection) if H is nonempty and closed under taking subsets

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- $\mathrm{rk}(V, H) = \max\{|X| : X \in H\}$
- Graphs are simplicial complexes of rank 2
- Matroids are simplicial complexes satisfying

(EP) For all $I, J \in H$ with $|I| = |J| + 1$, there exists some $i \in I \setminus J$ such that $J \cup \{i\} \in H$.

The superboolean semiring

$\mathbb{SB} = \{0, 1, 1^\nu\}$

$+$	0	1	1^ν
0	0	1	1^ν
1	1	1^ν	1^ν
1^ν	1^ν	1^ν	1^ν

$\cdot$	0	1	1^ν
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- The vectors $C_1, \ldots, C_m \in \mathbb{SB}^n$ are dependent if $\lambda_1 C_1 + \ldots \lambda_m C_m \in \{0, 1^\nu\}$ for some $\lambda_1, \ldots, \lambda_m \in \{0, 1\}$ not all zero
- The permanent is the positive version of the determinant

Superboolean matrices

Proposition (Izhakian and Rhodes 2011)

The following conditions are equivalent for every $M \in \mathcal{M}_n(\mathbb{SB})$:

(i) the column vectors of M are independent;

(ii) $\mathrm{Per}\, M = 1$;

(iii) M can be transformed into some lower triangular matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ ? & 1 & 0 & \dots & 0 \\ ? & ? & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ? & ? & ? & \dots & 1 \end{pmatrix}$$

by permuting rows and permuting columns independently.

Rank of a matrix

A square matrix with permanent 1 is nonsingular.

Proposition (Izhakian 2006)

The following are equal for a given $m \times n$ superboolean matrix M:

(i) the maximum number of independent column vectors in M;

(ii) the maximum number of independent row vectors in M;

(iii) the maximum size of a nonsingular submatrix of M.

This number is the rank of M.

Representation of $\vee$-generated lattices

If X $\vee$-generates the lattice L, let $M(L,X) = (m_{\ell,x})$ be the boolean $L \times X$ matrix defined by

$$m_{\ell,x} = \begin{cases} 0 & \text{if } x \leq \ell \\ 1 & \text{otherwise} \end{cases}$$

Proposition

(i) The column subset $X' \subseteq X$ is independent if and only if it admits an enumeration $x_1, \ldots, x_k$ such that

$$x_1 < (x_1 \vee x_2) < \ldots < (x_1 \vee \ldots \vee x_k).$$

(ii) The rank of $M(L,X)$ equals the height of the lattice L.

Graphs

The boolean representation

- Let $\Gamma = (V, E)$ be a finite graph with $V = \{1, \ldots, n\}$.
- The adjacency matrix of Γ is the $n \times n$ boolean matrix $A_\Gamma = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

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- But we shall prefer the matrix A_Γ^c obtained by interchanging 0 and 1 all over A_Γ.

The lattice of stars

- If $\Gamma = (V, E)$ and $v \in V$, let $\mathrm{St}(v)$ be the set of vertices adjacent to v
- If $W \subseteq V$, let $\mathrm{St}(W) = \cap_{w \in W} \mathrm{St}(w)$

The lattice of stars

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- If $W \subseteq V$, let $\mathrm{St}(W) = \cap_{w \in W} \mathrm{St}(w)$
- $\mathrm{St}\,\Gamma = \{\mathrm{St}(W) \mid W \subseteq V\}$ ordered by inclusion is a lattice (with intersection as meet, and determined join)
- $\{y_1, \ldots, y_k\}$ is a transversal of the partition of the successive differences for the chain $X_0 \supset \ldots \supset X_k$ if $y_i \in X_{i-1} \setminus X_i$ for $i = 1, \ldots, k$.

Matrices versus lattices

Theorem

Given a finite graph $\Gamma = (V, E)$ and $W \subseteq V$, the following conditions are equivalent:

(i) the column vectors $A^c[w]$ $(w \in W)$ are independent;

(ii) W is a transversal of the partition of successive differences for some chain of $\mathrm{St}\,\Gamma$.

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The height of a lattice L is the length of the longest chain in L.

Theorem

Let $\Gamma = (V, E)$ be a finite graph. Then $\mathrm{rk}\, A^c_\Gamma = \mathrm{ht}\, \mathrm{St}\, \Gamma$.

Partial euclidean geometries

Let P be a finite nonempty set (points) and let $\mathcal{L}$ be a nonempty subset of 2^P (lines). We say that $(P, \mathcal{L})$ is a PEG if:

(P1) $P \subseteq \cup\mathcal{L}$;

(P2) if $L, L' \in \mathcal{L}$ are distinct, then $|L \cap L'| \leq 1$;

(P3) $|L| \geq 2$ for every $L \in \mathcal{L}$.

Graphs and Coxeter's configurations are particular cases of PEGs.

From graphs to PEGs

- A graph is sober if $\mathrm{St}|_V$ is injective
- Every graph admits a retraction onto a sober connected restriction with the same lattice of stars
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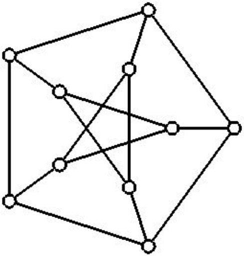
- Every graph admits a retraction onto a sober connected restriction with the same lattice of stars
- The class of sober connected graphs of rank 3 (SC3) contains all cubic graphs of girth ≥ 5 and has many interesting features
- Given a graph $\Gamma = (V, E)$, let $\mathcal{L}_\Gamma = \{W \in \mathrm{St}\,\Gamma \setminus \{V\} : |W| \geq 2\}$ and let $\mathrm{Geo}\,\Gamma = (V, \mathcal{L}_\Gamma)$

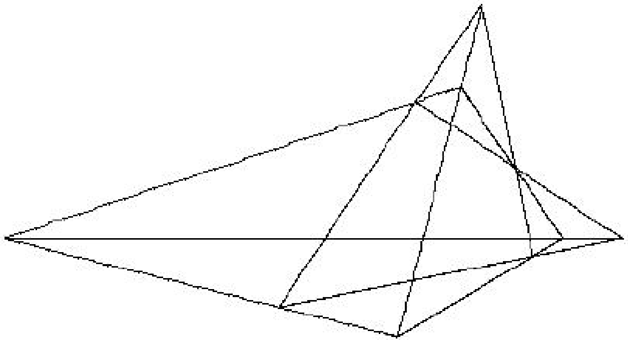
Theorem

If $\Gamma \in \mathrm{SC3}$, then $\mathrm{Geo}\,\Gamma$ is a PEG.

Starting with the Petersen graph. . .



. . . we get the Desargues configuration!



PEGs, graphs and lattices

- In the dual of a PEG, lines become the points
- The Levi graph of a PEG $(P, \mathcal{L})$ has $P \cup \mathcal{L}$ as vertex set and all the natural edges between points and lines

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Theorem

Let $\mathcal{G}$ and $\mathcal{G}'$ be PEG's with $\operatorname{mindeg} \mathcal{G}, \operatorname{mindeg} \mathcal{G}' \geq 2$. Then the following conditions are equivalent:

(i) $\mathcal{G} \cong \mathcal{G}'$ or $\mathcal{G}^d \cong \mathcal{G}'$;

(ii) $\operatorname{Levi} \mathcal{G} \cong \operatorname{Levi} \mathcal{G}'$;

(iii) $\operatorname{St} \operatorname{Levi} \mathcal{G} \cong \operatorname{St} \operatorname{Levi} \mathcal{G}'$.

Simplicial complexes

Boolean representations

- A simplicial complex (V, H) is boolean representable if there exists some $R \times V$ boolean matrix M such that

$$X \in H \quad \Leftrightarrow \quad \text{the column vectors } M[x] \ (x \in X) \text{ are independent over } \mathbb{SB}$$

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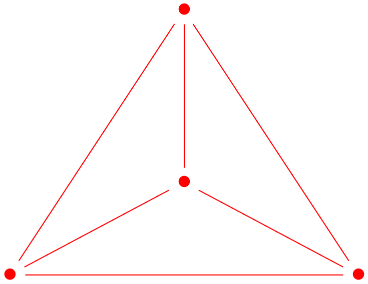
  holds for every $X \subseteq V$
- The representation is reduced if all rows are distinct
- All matroids are boolean representable (Izhakian and Rhodes 2011), unlike field representable
- Not all simplicial complexes are boolean representable

Example: tetrahedra

The nature of the simplicial complex having K_4 as its 2-skeleton depends on the number of 3-faces:



- 0, 3 or 4 3-faces: matroid, hence boolean representable
- 2 3-faces: not a matroid, but boolean representable
- 1 3-face: not boolean representable

Flats

- $X \subseteq V$ is a flat if

$$\forall I \in H \cap 2^X \; \forall v \in V \setminus X \qquad I \cup \{v\} \in H$$

- The set of all flats of (V, H) is denoted by $\mathrm{Fl}(V, H)$

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- $\mathrm{Fl}(V, H)$ ordered by inclusion is a lattice (with intersection as meet, and determined join)

The canonical representation

$M(\mathrm{Fl}(V,H)) = (m_{Fv})$ is the $\mathrm{Fl}(V,H) \times V$ matrix defined by

$$m_{Fv} = \begin{cases} 0 & \text{if } v \in F \\ 1 & \text{otherwise} \end{cases}$$

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Theorem

Let (V,H) be a simple simplicial complex. Then the following conditions are equivalent:

(i) (V,H) is boolean representable;

(ii) $M(\mathrm{Fl}(V,H))$ is a reduced boolean representation of (V,H).

Moreover, in this case any other reduced boolean representation of (V,H) is congruent to a submatrix of $M(\mathrm{Fl}(V,H))$.

The lattice of boolean representations

- These submatrices correspond to certain $\cap$-subsemilattices of $\mathrm{Fl}(V, H)$
- This helps to define a lattice structure on the set of boolean representations of (V, H)

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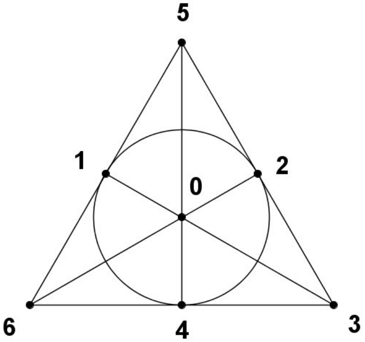
- This helps to define a lattice structure on the set of boolean representations of (V, H)
- In this lattice, the strictly join irreducible representations deserve special attention, and among these the minimal representations

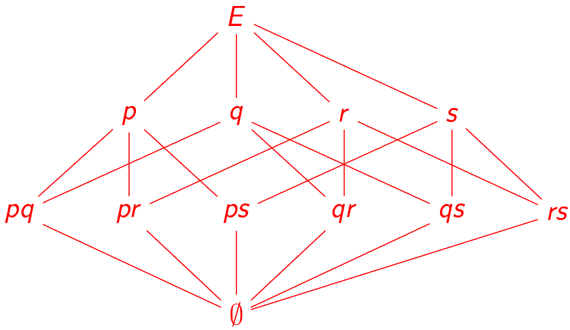
The Fano matroid

We take $V = \{0, \ldots, 6\}$ and (V, H) of rank 3 by excluding the 7 lines in the Fano plane (the projective plane of order 2 over $\mathbb{F}_2$:



Minimal representations: lattices

The flats are $\emptyset, V$, the points and the 7 lines. We obtain lattices of the form below (where p, q, r, s are lines and $pq = p \cap q$):



Minimal representations: matrices

...which can be realized by matrices of the form:

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Shelling

- A basis of a simplicial complex (V, H) is a maximal element of H
- If all the bases have the same cardinal (such as in matroids), (V, H) is pure

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- A basis of a simplicial complex (V, H) is a maximal element of H
- If all the bases have the same cardinal (such as in matroids), (V, H) is pure
- (V, H) is shellable if we can order its bases as $B_1, \ldots, B_t$ so that, for $I(B_k) = (\cup_{i=1}^{k-1} 2^{B_i}) \cap 2^{B_k}$,

  $$(B_k, I(B_k)) \text{ is pure of rank } |B_k| - 1$$

  for $k = 2, \ldots, t$
- Such an ordering is called a shelling

Geometric realization

- Every (abstract) simplicial complex (V, H) admits an euclidean geometric realization, denoted by $||(V, H)||$
- The topological space $||(V, H)||$ is unique up to homeomorphism

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- Every (abstract) simplicial complex (V, H) admits an euclidean geometric realization, denoted by $||(V, H)||$
- The topological space $||(V, H)||$ is unique up to homeomorphism
- A wedge of mutually disjoint connected topological spaces X_i is obtained by selecting a base point $x_i \in X_i$ and then identifying all the x_i
- If $B_1, \ldots, B_t$ is a shelling of (V, H), we say that B_k $(k > 1)$ is a homology basis in this shelling if $2^{B_k} \setminus \{B_k\} \subseteq \cup_{i=1}^{k-1} 2^{B_i}$.

Geometric perspective of shellability

Theorem (Björner and Wachs (1996))

Let (V, H) be a shellable simplicial complex of rank r. Then:

(i) $||(V, H)||$ has the homotopy type of a wedge $W(V, H)$ of spheres of dimensions from 1 to $r - 1$;

(ii) for $i = 1, \ldots, r - 1$, the number $\beta_i(V, H)$ of i-spheres in the construction of $W(V, H)$ is the number of homology $(i + 1)$-bases in a shelling of (V, H).

Indeed, $\beta_i(V, H)$ is the ith Betti number of the topological space $||(V, H)||$.

The graph of flats

- To understand shellability for simple simplicial complexes of rank 3, we need the concept of graph of flats
- The graph of flats $\Gamma\mathrm{Fl}(V,H)$ has vertex set V and edges of the form v — w whenever $v \neq w$ and $v, w \in F$ for some $F \in \mathrm{Fl}(V,H) \setminus \{V\}$

Characterizing the graphs of flats

An anticlique is a totally disconnected subset of vertices.

Theorem

Let $\Gamma = (V, E)$ be a finite graph. Then $\Gamma \cong \Gamma\mathrm{Fl}(V, H)$ for some boolean representable simple (V, H) of rank 3 if and only if the following conditions are satisfied:

(i) $|V| \geq 3$;

(ii) $A \neq \emptyset$;

(iii) for every nontrivial anticlique X of Γ, there exists some 3-anticlique Y such that $|X \cap Y| = 2$.

The graph of flats determines shellability

Theorem

Let (V, H) be a boolean representable simple simplicial complex of rank 3. Then the following conditions are equivalent:

(i) (V, H) is shellable;

(ii) $\Gamma\mathrm{Fl}(V, H)$ has at most 2 connected components or at most 1 nontrivial connected component.

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We have also obtained formulae to compute the Betti numbers $\beta_i(V, H)$.