Problem 1. Let \( \vec{v}_1 = (1, 2, 3)^T, \vec{v}_2 = (1, 1, -1)^T, \vec{v}_3 = (0, -1, -2)^T. \)
(a) Show that these vectors form a basis in \( \mathbb{R}^3 \).

Solution. The determinant of the matrix \( A \) whose columns are \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) is not zero, so the columns are linearly independent. Since the dimension of \( \mathbb{R}^3 \) is 3, these vectors form a basis of \( \mathbb{R}^3 \).

(b) Find the transition matrix from the standard basis to that basis.

Solution. Let \( \mathcal{B} \) be that basis. Then the matrix \( A \) is the transition matrix \([\mathcal{B} \to \mathcal{S}].\)

(c) Use the transition matrix to find coordinates of the vector \((2, 3, 1)^T\) in that basis.

Problem 2. How many solutions will the linear system \( Ax = b \) have
(a) if \( b \) is in the column space of \( A \) and the columns of \( A \) are linearly independent?
(b) if \( b \) is not in the column space of \( A \)
Explain your answer.

Solution. (a) Since \( b \) is in the column space of \( A \), it is a linear combination of columns of \( A \), hence there is a solution of the system \( Ax = b \). Since the columns of \( A \) are linearly independent in the reduced row echelon form of \( A \) every column will have a pivot. Therefore the system \( Ax = b \) does not have free unknowns, hence it has exactly one solution.

(b) Since \( b \) is not in the column space of \( A \), it is not a linear combination of columns of \( A \), hence \( Ax = b \) has no solutions.

Problem 3. Find the matrix of the projection of \( \mathbb{R}^3 \) onto the the plane \( x - y - z = 0 \).

Solution. A normal vector of the plane is \( \vec{n} = (1, -1, -1)^T \). The vectors \( \vec{a} = (1, 1, 0)^T, \) \( \vec{b} = (1, 0, 1)^T \) are parallel to the plane. These three vectors \( \vec{n}, \vec{a}, \vec{b} \) form a basis \( \mathcal{B} \) of \( \mathbb{R}^3 \). Let \([\mathcal{B} \to \mathcal{S}]\) be the corresponding transition matrix. In the basis \( \mathcal{B} \), the matrix of the projection is \( P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) (because the projection of \( \vec{n} \) is \( \vec{0} \), the projection of \( \vec{a} \) is \( \vec{a} \),

the projection of \( \vec{b} \) is \( \vec{b} \). Then the matrix of the projection is \([\mathcal{B} \to \mathcal{S}]P[\mathcal{B} \to \mathcal{S}]^{-1}\).

Problem 4. (a) Is it possible for a matrix to have the vector \((1, 2, 3)\) in its row space and the vector \((2, -1, -1)^T\) in its null space?

Solution. No, it is not possible. If a vector \( \vec{a} \) in the row space of a matrix, then it is a linear combination \( \alpha_1 \vec{r}_1 + \ldots + \alpha_n \vec{r}_n \) of the rows of the matrix. If \( \vec{b}^T \) is in the null space of the matrix, then \( \vec{r}_i \vec{b}^T = 0 \) for each \( i \). Therefore \( \vec{a} \vec{b}^T \) should be equal to 0, but \((1, 2, 3)/(2, -1, -1)^T = -3 \neq 0 \).

(b) Give an example of a matrix having \((1, 2, 3)\) in its row space and \((2, -1, 0)^T\) in its null space.

Solution. Let \( A \) be the \( 1 \times 3 \) matrix consisting of one row \((1, 2, 3)\). Then \( A(2, -1, 0)^T = 0 \), hence \((2, -1, 0)^T\) is in the null space of \( A \). Clearly, \((1, 2, 3)\) is in the row space of \( A \).

Problem 5. Find the matrix of the linear transformation of \( \mathbb{R}^2 \) which is the composition of dilation by 1/2, rotation through 60 degrees and reflection about the line \( y = x \).
**Solution.** The basic vectors \((1, 0)\), and \((0, 1)\) are mapped by this linear transformation as follows (we first apply the dilation, then the rotation, then the reflection):

\[
(1, 0)^T \rightarrow \left(\frac{\sqrt{3}}{4}, 0\right) \rightarrow \left(\frac{1}{4}, -\frac{\sqrt{3}}{4}\right)
\]

\[
(0, 1)^T \rightarrow \left(0, \frac{1}{2}\right)^T \rightarrow \left(\frac{1}{4}, -\frac{\sqrt{3}}{4}\right)
\]

Therefore the matrix of this linear transformation is

\[
\begin{pmatrix}
\frac{\sqrt{3}}{4} & \frac{1}{4} \\
\frac{1}{4} & -\frac{\sqrt{3}}{4}
\end{pmatrix}
\]

**Problem 6.** Prove that if \(A, B, C\) are square matrices of the same size, and \(A\) is similar to \(B\), \(B\) is similar to \(C\), then \(A\) is similar to \(C\).

**Solution.** Since \(A\) is similar to \(B\), there exists a matrix \(S\) such that \(A = S^{-1}BS\). Since \(B\) is similar to \(C\), there exists a matrix \(T\) such that \(B = T^{-1}CT\). Therefore \(A = S^{-1}T^{-1}CTS = (TS)^{-1}C(TS)\). Thus \(A\) is similar to \(C\).

**Problem 7.** Consider functions \(1 + x^2, x - 1, x^2 + x + 1\) as elements of the vector space \(C[0, 1]\). Are these functions linearly independent? Explain your answer.

**Solution.** The Wronskian of these functions is the determinant of the matrix

\[
\begin{pmatrix}
1 + x^2 & x - 1 & x^2 + x + 1 \\
2x & 1 & 2x + 1 \\
2 & 0 & 2
\end{pmatrix}
\]

which is equal to \(-2\). Since it is not equal to 0, the functions are linearly independent.