Definition
Let $K$ and $L$ be finite CW complexes. There is an elementary expansion from $K$ to $L$ if $L = K \cup_f D^n$ where $f : D^{n-1} \to K$. We say that there is an elementary collapse from $L$ to $K$. A homotopy equivalence is called simple if it is homotopic to a homotopy equivalence induced by a sequence of elementary expansions and collapses.

Theorem
If $K$ is simply connected, then every homotopy equivalence $K \to L$ is simple.
Simple-homotopy

Example
There are finite complexes with $\pi_1 = \mathbb{Z}_5$ that are homotopy equivalent but not simple-homotopy equivalent.

$K$ is obtained by attaching $D^2$ to $S^1$ using a map of degree 5. $L$ is obtained from $K$ by wedging with $S^2$ and then attaching $D^3$ according to the prescription $1 - t + t^3$. \(^1\)

In general, if $K$ and $L$ have fundamental group $\pi$ and $f : K \to L$ is a homotopy equivalence, $f$ is simple if and only if a torsion, $\tau(f)$ in the Whitehead group $Wh(\pi)$ of $\pi$ is trivial. $Wh(\pi)$ measures whether an invertible matrix with entries in the integral group ring $\mathbb{Z}\pi$ can be row and column reduced to a matrix with $\pm$ group elements along the diagonal.

\[^1\](1 - t + t^2)(t + t^2 - t^4) = 1
Question: If $K$ and $L$ are homeomorphic simplicial complexes, must $K$ and $L$ be piecewise-linear(ly?) homeomorphic?

Originally, this was thought of as an approach to proving the topological invariance of simplicial homology. Of course, the introduction of the notion of homotopy equivalence gave a much easier proof of a much stronger theorem.

**Theorem (Milnor)**

*There exist finite simplicial complexes $K$ and $L$ that are homeomorphic but that are not PL homeomorphic.*
Theorem (Chapman)

If \( f : K \rightarrow L \) is a homeomorphism between simplicial complexes, then \( \tau(f) = 0 \).

Chapman’s proof was modeled on Kirby-Siebenmann’s work on the Hauptvermutung for PL manifolds, but in the setting of Hilbert cube manifolds.
Topological invariance of torsion

Definition
We call a homotopy equivalence $f : K \to L$ between simplicial complexes an $\epsilon$-equivalence if there exist a homotopy equivalence $g : L \to K$ and homotopies $h_t : f \circ g \simeq id$ and $k_t : g \circ f \simeq id$ so that $\text{diam}\{h_t(x) | 0 \leq t \leq 1\} < \epsilon$ for each $x \in L$ and $\text{diam}\{f(k_t(y)) | 0 \leq t \leq 1\} < \epsilon$ for each $y \in K$.

Theorem (F.)
Given $L$, there is an $\epsilon > 0$ so that if $f : K \to L$ is an $\epsilon$-equivalence, then $\tau(f) = 0$.

The first proof of this showed that $K \times Q$ and $L \times Q$ were homeomorphic, $Q$ being the Hilbert cube, whence the result followed from Chapman. However, this point of view soon led to more direct proofs of Chapman’s theorem.
Topological manifolds, $n \geq 5$

Theorem (Chapman-F.)

If $M^n$ is a closed connected topological manifold, $n \geq 5$, then given $\epsilon > 0$, there is a $\delta > 0$ so that if $f : N \rightarrow M$ is an $\delta$-equivalence, $N$ closed, then $f$ is $\epsilon$-homotopic to a homeomorphism.

Due to the efforts of Freedman-Quinn, Perlman, and others, this result is now known in all dimensions.

Theorem (F)

If $M^n$ is a closed connected topological manifold, $n \geq 5$, then there is an $\epsilon > 0$ so that if $f : M \rightarrow N$ is a map to a connected manifold of the same dimension such that $\text{diam } f^{-1}(x) < \epsilon$ for each $x \in N$, then $f$ is homotopic to a homeomorphism.
Grove-Petersen-Wu

Question: Do such homotopy equivalences occur naturally? (Yes, in geometric topology, but I’ll give an application to differential geometry.)

Theorem (Grove-Petersen-Wu)

The collection of closed Riemannian n-manifolds, $n \geq 5$, with diameter $< D$, volume $> v$, and sectional curvature $> \kappa$ contains only finitely many homeomorphism (and therefore diffeomorphism) types.

As above, this result is now known for homeomorphisms in all dimensions. Some of Perlman’s work generalizes Grove-Petersen-Wu. This example is included to give a general idea of what applications might look like.

Definition

A monotone function $\rho : [0, R) \to [0, \infty)$ is a contractibility function for a space $X$ if $B_t(x)$ contracts to a point in $B_{\rho(t)}(x)$ for every $x \in X$ and $t \in [0, R]$, $\rho(0) = 0$ and $\rho(t) \geq t$. 
Theorem (Grove-Petersen)

There is a function $\rho : [0, R) \to [0, \infty)$ which is a contractibility function for every closed Riemannian $n$-manifold, $n \geq 5$, with diameter $< D$, volume $> v$, and sectional curvature $> \kappa$.

This collection of Riemannian manifolds is precompact in Gromov-Hausdorff space. It is easy to see that manifolds with contractibility function $\rho$ that are close enough together must be epsilon homotopy equivalent. Therefore, if limit points of the collection are manifolds, we’re done. An argument involving crossing with a two-torus and peeling the factors off again removes this last hurdle.
Theorem (Dranishnikov-F. flexibility)

There exist Riemannian manifolds $M_t$ and $N_t$, $0 \leq t < 1$ and a function $\rho : [0, R) \to [0, \infty)$ which is a contractibility function for each $M_t$ and $N_t$, so that $\lim_{t \to 1} M_t = \lim_{t \to 1} N_t$ with $M_t$’s homeomorphic to each other, $N_t$’s homeomorphic to each other, but $M_t$’s not homeomorphic to $N_t$’s. These manifolds do have the same simple-homotopy types and the same rational Pontrjagin classes.

Theorem (Dranishnikov-F. rigidity)

If $M_t$ is two-connected and the homology of $M_t$ contains no odd torsion, then the phenomenon above can’t happen.

The difference between this and the situation that Grove-Petersen encountered is that the common limit of the $M_t$’s and $N_t$’s can be infinite-dimensional and in this case the homeomorphism type can vary, but only by finitely many homeomorphism types.
Theorem (Dranishnikov-F.)

If $\mathcal{C}$ is a precompact collection of Riemannian $n$-manifolds, $n \neq 3$, such that there is a contractibility function $\rho : [0, R) \rightarrow [0, \infty)$ which is a contractibility function for each $M \in \mathcal{C}$, then $\mathcal{C}$ contains only finitely many homeomorphism types.

Actually, this theorem appears earlier in a paper of mine in the Duke Journal. The argument in the paper with Dranishnikov is different and more illuminating.