Bratteli Diagrams and the Unitary Duals of Locally Finite Groups

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Unitary Representations of Discrete Groups

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Definition

Two representations $\varphi : G \to U(\mathcal{H})$ *and* $\psi : G \to U(\mathcal{H})$ *are* ***unitarily equivalent*** *if there exists* $A \in U(\mathcal{H})$ *such that*

$$\psi(g) = A\,\varphi(g)\,A^{-1} \quad \text{for all } g \in G.$$

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Definition

The unitary representation $\varphi : G \to U(\mathcal{H})$ is irreducible if there are no nontrivial proper G-invariant closed subspaces $0 < W < \mathcal{H}$.

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- The irreducible unitary representations of $\mathbb{Z}$ are

$$\varphi_z : \mathbb{Z} \to U_1(\mathbb{C}) = \mathbb{T} = \{ c \in \mathbb{C} : |c| = 1 \}$$

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- The multiplicity-free unitary representations of $\mathbb{Z}$ can be parameterized by the Borel probability measures μ on $\mathbb{T}$ so that the following are equivalent:
  (i) the representations φ_μ, φ_ν are unitarily equivalent;
  (ii) the measures μ, ν have the same null sets.

The Polish Space of Unitary Representations

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- The set $\mathrm{Rep}(G) \subseteq U(\mathcal{H})^G$ of unitary representations is a closed subspace and hence $\mathrm{Rep}(G)$ is a Polish space.
- The set $\mathrm{Irr}(G)$ of irreducible representations is a G_δ subset of $\mathrm{Rep}(G)$ and hence $\mathrm{Irr}(G)$ is also a Polish space.

Borel equivalence relations

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An equivalence relation E on a Polish space X is Borel if E is a Borel subset of $X \times X$.

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Theorem (Mackey)

The unitary equivalence relation $\approx_G$ on $\mathrm{Irr}(G)$ is an F_σ equivalence relation.

Theorem (Hjorth-Törnquist)

The unitary equivalence relation $\approx_G^+$ on $\mathrm{Rep}(G)$ is an $F_{\sigma\delta}$ equivalence relation.

Smooth vs Nonsmooth

Definition (Mackey)

The Borel equivalence relation E on the Polish space X is *smooth* *if there exists a Borel map $f : X \to \mathbb{R}$ such that*

$$x \, E \, y \iff f(x) = f(y).$$

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Corollary

If G is a countable group, then unitary equivalence for finite dimensional irreducible unitary representations of G is smooth.

The Glimm-Thoma Theorem

Theorem (Glimm-Thoma)

If G is a countable group, then the following are equivalent:

(i) *G is not abelian-by-finite.*

(ii) *G has an infinite dimensional irreducible representation.*

(iii) *The unitary equivalence relation $\equiv_G$ on the space $\mathrm{Irr}(G)$ of infinite dimensional irreducible unitary representations of G is not smooth.*

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Question

Does this mean that we should abandon all hope of finding a "satisfactory classification" for the irreducible unitary representations of the other countable groups?

Borel reductions

Definition (Friedman-Kechris)

Let E, F be Borel equivalence relations on the Polish spaces X, Y.

- $E \leq_B F$ *if there exists a Borel map* $\varphi : X \to Y$ *such that*

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In this case, f is called a Borel reduction from E to F.

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  In this case, f is called a ***Borel reduction*** *from E to F.*
- $E \sim_B F$ *if both* $E \leq_B F$ *and* $F \leq_B E$.
- $E <_B F$ *if both* $E \leq_B F$ *and* $E \nsim_B F$.

The Glimm-Effros Dichotomy

Theorem (Harrington-Kechris-Louveau)

If E is a Borel equivalence relation on the Polish space X, then exactly one of the following holds:

(i) *E is smooth; or*

(ii) $E_0 \leq_B E$.

Definition

E_0 is the Borel equivalence relation on $2^{\mathbb{N}}$ defined by:

$$x \, E_0 \, y \quad \Longleftrightarrow \quad x_n = y_n \quad \text{for all but finitely many } n.$$

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Example

Baer's classification of the rank 1 torsion-free abelian groups is essentially a Borel reduction to E_0.

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Theorem (Hjorth 1997)

If the countable group G is not abelian-by-finite , then there exists a $U(\mathcal{H})$-invariant Borel subset $X \subseteq \mathrm{Irr}(G)$ such that the unitary equivalence relation $\approx_G \restriction X$ is turbulent.

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Remark

This is a much more serious obstruction to the existence of a "satisfactory classification" of the irreducible unitary representations of G.

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Question (Dixmier-Effros-Thomas)

Do there exist countable groups G, H such that

(i) *G, H are not abelian-by-finite; and*

(ii) *$\approx_G$, $\approx_H$ are not Borel bireducible?*

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Conjecture (Thomas)

If G is a nonabelian free group and H is a "suitably chosen" amenable group, then $\approx_H <_B \approx_G$.

Nonabelian free groups

Notation

$\mathbb{F}_n$ denotes the free group on n generators for $n \in \mathbb{N}^+ \cup \{ \infty \}$.

Observation

If G is any countable group, then $\approx_G$ is Borel reducible to $\approx_{\mathbb{F}_\infty}$.

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If G is any countable group, then $\approx_G$ is Borel reducible to $\approx_{\mathbb{F}_\infty}$.

Proof.

If $\theta : \mathbb{F}_\infty \to G$ is a surjective homomorphism, then the induced map

$$\mathrm{Irr}(G) \to \mathrm{Irr}(\mathbb{F}_\infty)$$
$$\varphi \mapsto \varphi \circ \theta$$

is a Borel reduction from $\approx_G$ to $\approx_{\mathbb{F}_\infty}$. □

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Sketch Proof.

If $f : \mathbb{N} \to \mathbb{N}$ be a suitably fast growing function, then we can induce representations from

$$\mathbb{F}_\infty = \langle\, a^{f(n)}\, b\, a^{-f(n)} \mid n \in \mathbb{N} \,\rangle \leqslant N = \langle\, a^m\, b\, a^{-m} \mid m \in \mathbb{N} \,\rangle$$

to the free group $\mathbb{F}_2 = \langle\, a, b \,\rangle$. □

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- *Does $H \leqslant G$ imply that $\approx_H$ is Borel reducible to $\approx_G$?*

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Question

- *Does $H \leqslant G$ imply that $\approx_H$ is Borel reducible to $\approx_G$?*
- *In particular, is $\approx_{\mathbb{F}_2}$ Borel reducible to $\approx_{SL(3,\mathbb{Z})}$?*

A suitably chosen amenable group?

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A countable group G is amenable if there exists a left-invariant finitely additive probability measure $\mu : \mathcal{P}(G) \to [\,0, 1\,]$.

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Some Candidates?

- *The direct sum* $\bigoplus_{n \in \mathbb{N}} \mathrm{Sym}(3)$ *of countably many copies of* $\mathrm{Sym}(3)$*.*
- *A countably infinite extra-special p-group P; i.e.* $P' = Z(P)$ *is cyclic of order p and* $P/Z(P)$ *is elementary abelian p-group.*

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- The following result is an immediate consequence of the work of Glimm (1961) and Elliot (1977).

Theorem

Let H be a countable locally finite group. If the countable group G is not abelian-by-finite, then $\approx_H$ is Borel reducible to $\approx_G$.

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- The following result is an immediate consequence of the work of Glimm (1961) and Elliot (1977).

Theorem

Let H be a countable locally finite group. If the countable group G is not abelian-by-finite, then $\approx_H$ is Borel reducible to $\approx_G$.

Corollary

If G, H are countable locally finite groups, neither of which is abelian-by-finite, then $\approx_G$ and $\approx_H$ are Borel bireducible.

The reduced C^*-algebra

Definition

If G is a countably infinite group, then the left regular representation

$$\lambda : G \to U(\,\ell^2(G)\,)$$

extends to an injective $*$-homomorphism of the group algebra

$$\lambda : \mathbb{C}[\,G\,] \to \mathcal{L}(\,\ell^2(G)\,).$$

The reduced C^*-algebra $C^*_\lambda(\,G\,)$ is the completion of $\mathbb{C}[\,G\,]$ with respect to the norm $||x||_r = ||\lambda(x)||_{\mathcal{L}(\,\ell^2(G)\,)}$.

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Remark

If G is amenable, then there is a canonical correspondence between the irreducible representations of G and $C^*_\lambda(\,G\,)$.

Approximately finite dimensional C^*-algebras

Definition

A C^*-algebra A is approximately finite dimensional if $A = \overline{\bigcup_{n \in \mathbb{N}} A_n}$ is the closure of an increasing chain of finite dimensional sub-C^*-algebras A_n.

Example

If $G = \bigcup_{n \in \mathbb{N}} G_n$ is a locally finite group, then $C^*_\lambda(G) = \overline{\bigcup_{n \in \mathbb{N}} \mathbb{C}[G_n]}$ is approximately finite dimensional.

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Remark

Every finite dimensional C^*-algebra is isomorphic to a direct sum

$$\mathrm{Mat}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathrm{Mat}_{n_t}(\mathbb{C})$$

of full matrix algebras.

Bratteli Diagrams

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Theorem

If $G = \bigcup_{n \in \mathbb{N}} G_n$ is a locally finite group, then the following are equivalent:

(i) *G is not abelian-by-finite.*

(ii) *There exists a subsequence $(\ell_n \mid n \in \mathbb{N})$ and irreducible representations $\pi_n \in Irr(G_{\ell_n})$ such that for all $n \in \mathbb{N}$, $(\pi_n, \pi_{n+1} \restriction G_{\ell_n}) \geq 2$.*

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Question

Is there an "elementary" proof of this result?

Elliot's Theorem

- Extending Glimm's Theorem, Elliot proved:

Theorem (Elliot 1977)

If $\mathcal{A}$ is an approximately finite-dimensional C^*-algebra and $\mathcal{B}$ is a separable C^*-algebra such that $\approx_{\mathcal{B}}$ is non-smooth, then $\approx_{\mathcal{A}}$ is Borel reducible to $\approx_{\mathcal{B}}$.

Corollary (Elliot 1977)

If $\mathcal{A}$, $\mathcal{B}$ are approximately finite-dimensional C^*-algebras such that $\approx_{\mathcal{A}}$, $\approx_{\mathcal{B}}$ are non-smooth, then $\approx_{\mathcal{A}}$ and $\approx_{\mathcal{B}}$ are Borel bireducible.

Even less as expected ...

Theorem (Sutherland 1983)

Let $H = \bigoplus_{n \in \mathbb{N}} \mathrm{Sym}(3)$. If G is any countable amenable group, then $\approx_G$ is Borel reducible to $\approx_H$.

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Let $H = \bigoplus_{n \in \mathbb{N}} \mathrm{Sym}(3)$*. If* G *is* *any* *countable amenable group, then* $\approx_G$ *is Borel reducible to* $\approx_H$*.*

Corollary

If G*,* H *are countable amenable groups, neither of which is abelian-by-finite, then* $\approx_G$ *and* $\approx_H$ *are Borel bireducible.*

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Corollary

If G, H *are countable amenable groups, neither of which is abelian-by-finite, then* $\approx_G$ *and* $\approx_H$ *are Borel bireducible.*

Remark

The theorem ultimately depends upon the Ornstein-Weiss Theorem that if G, H are countable amenable groups, then any free ergodic measure-preserving actions of G, H are orbit equivalent.

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- Let $Z = \{ \xi, \xi^2 \}^{\mathbb{N}} \subseteq \widehat{A}$ and let μ be the usual product probability measure on Z.
- Then K acts freely and ergodically on (Z, μ)
- For each irreducible cocycle $\sigma : K \times Z \to U(\mathcal{H})$, there exists a corresponding irreducible representation

$$\pi_\sigma : H \to U(L^2(Z, \mathcal{H})).$$

Irreducible cocycles

Irreducible cocycles

- If $\alpha, \beta : K \times Z \to U(\mathcal{H})$ are cocycles, then $\mathrm{Hom}(\alpha, \beta)$ consists of the Borel maps $b : Z \to \mathcal{L}(\mathcal{H})$ such that for all $g \in K$,

$$\alpha(g,x)\, b(x) = b(g \cdot x)\, \beta(g,x) \qquad \mu\text{-a.e. } x \in Z.$$

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The heart of the matter

If $K' \curvearrowright (Z', \mu')$ is orbit equivalent to $K \curvearrowright (Z, \mu)$, then the "cocycle machinery" is isomorphic via a Borel map.

Coding representations in cocycles

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- Let $X = 2^{\Gamma}$ and let ν be the product probability measure on Z.
- Then the shift action of Γ on (X, ν) is (essentially) free and strongly mixing.
- For each irreducible representation $\varphi : G \to U(\mathcal{H})$, we can define an irreducible cocycle $\sigma_{\varphi} : (G \times Z) \times X \to U(\mathcal{H})$ by

$$\sigma_{\varphi}(g, z, x) = \varphi(g)$$

Summing up ...

Definition

Let $\mathrm{Irr}(E_0)$ *be the space of irreducible cocycles*

$$\sigma : K \times Z \to U(\mathcal{H})$$

and let $\approx_{E_0}$ *be the equivalence relation defined by*

$$\sigma \approx_{E_0} \tau \iff \mathrm{Hom}(\sigma, \tau) \neq 0.$$

Theorem

If the countable group G *is amenable but not abelian-by-finite, then the unitary equivalence relation* $\approx_G$ *is Borel bireducible with* $\approx_{E_0}$.

Summing up ...

Definition

Let Irr(E_∞) *be the space of irreducible cocycles*

$$\sigma : \mathbb{F}_2 \times 2^{\mathbb{F}_2} \to U(\mathcal{H})$$

and let $\approx_{E_\infty}$ *be the equivalence relation defined by*

$$\sigma \approx_{E_\infty} \tau \quad \Longleftrightarrow \quad \mathrm{Hom}(\sigma, \tau) \neq 0.$$

Theorem

The unitary equivalence relation $\approx_{\mathbb{F}_2}$ *is Borel bireducible with* $\approx_{E_\infty}$*.*

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If the countable group G is amenable but not abelian-by-finite, then the unitary equivalence relation $\approx_G$ is Borel bireducible with $\approx_{E_0}$.

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The unitary equivalence relation $\approx_{\mathbb{F}_2}$ is Borel bireducible with $\approx_{E_\infty}$.

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The Main Conjecture/Dream

$\approx_{E_\infty}$ *is not Borel reducible to* $\approx_{E_0}$.