Uniform Spanning Forests, the First ℓ^2-Betti Number, and Uniform Isoperimetric Inequalities

by Russell Lyons

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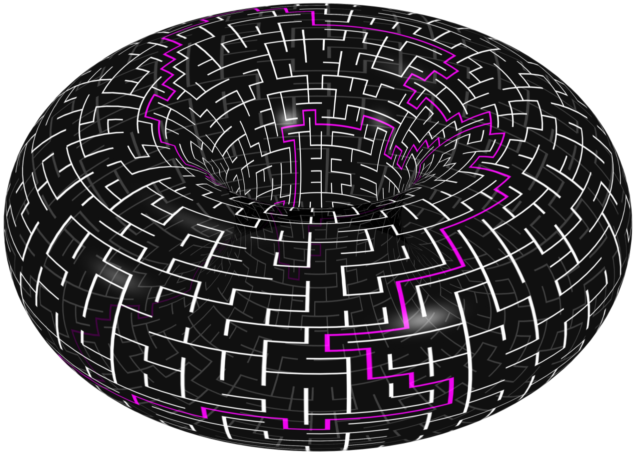
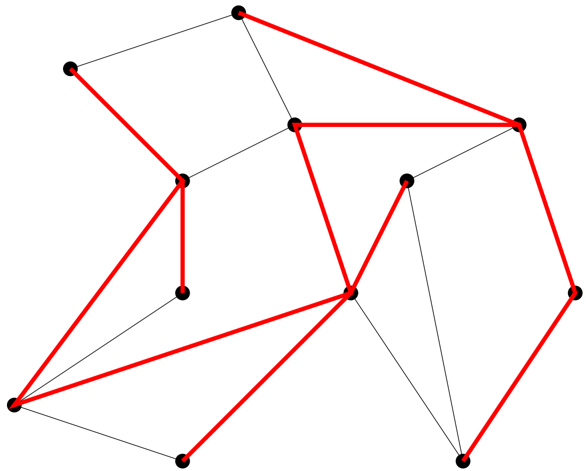
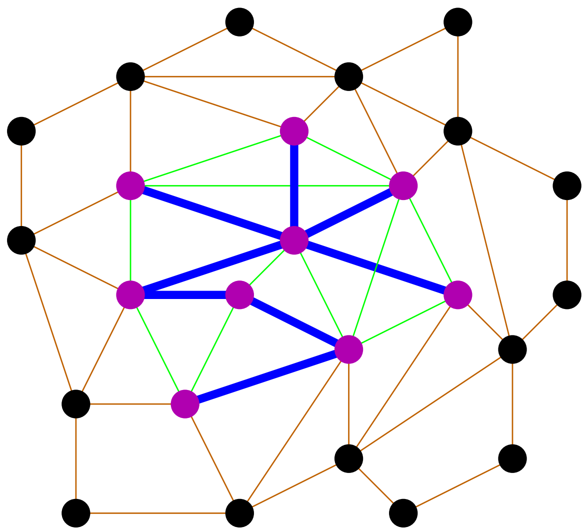


Uniform Spanning Trees

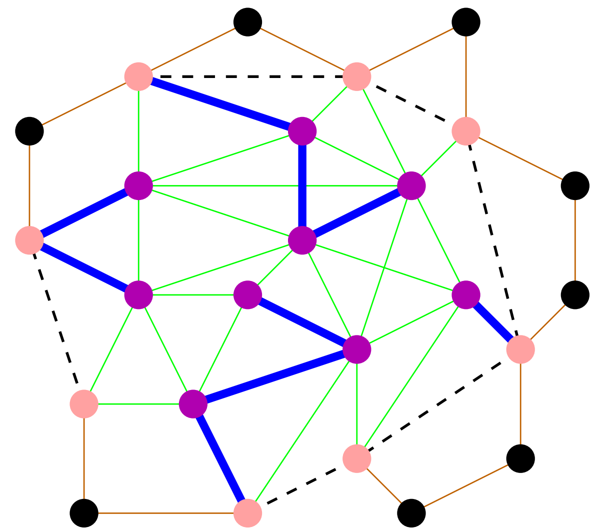


Algorithm of Aldous (1990) and Broder (1989): if you start a simple random walk at *any* vertex of a graph G and draw every edge it traverses except when it would complete a cycle (i.e., except when it arrives at a previously-visited vertex), then when no more edges can be added without creating a cycle, what will be drawn is a uniformly chosen spanning tree of G.

Infinite Graphs

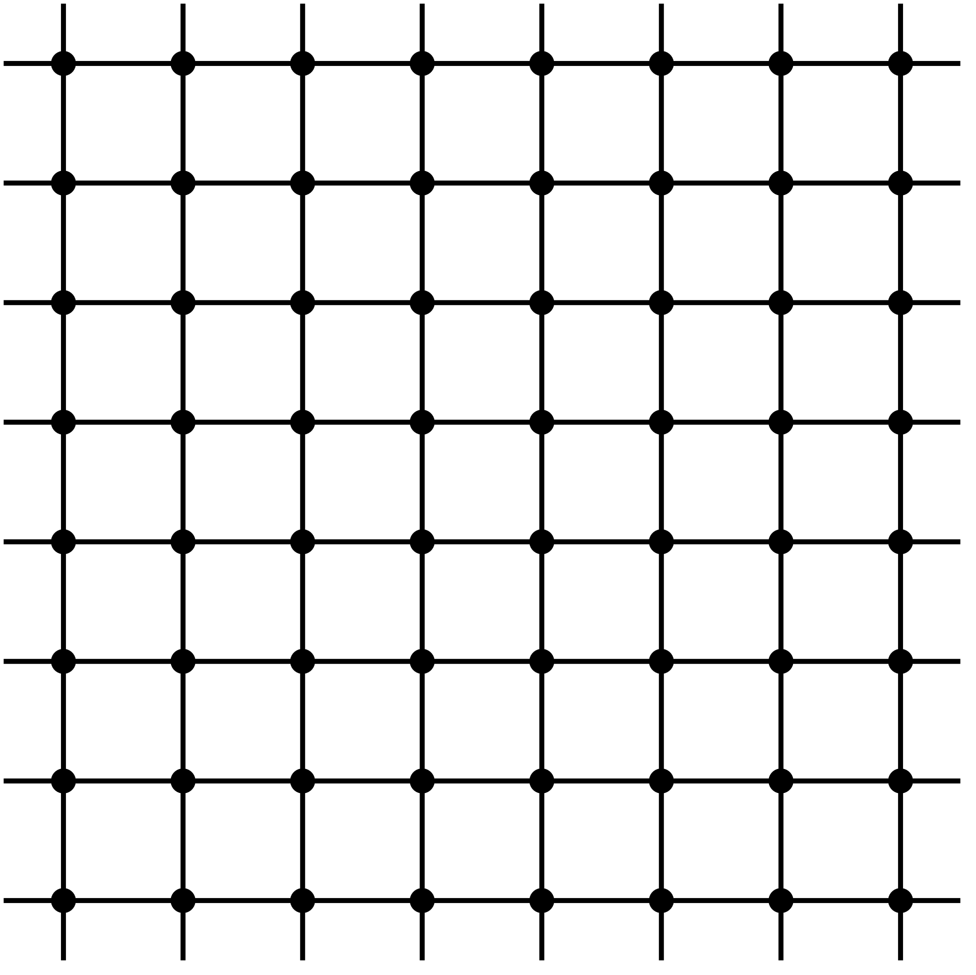


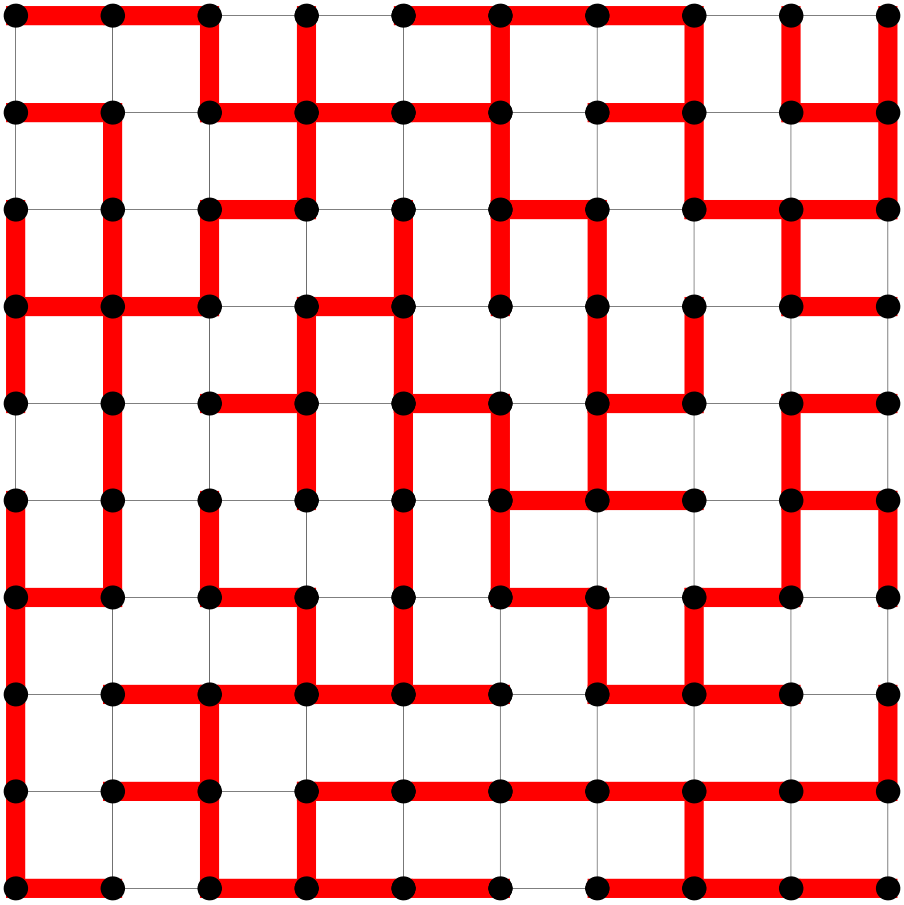
FUSF

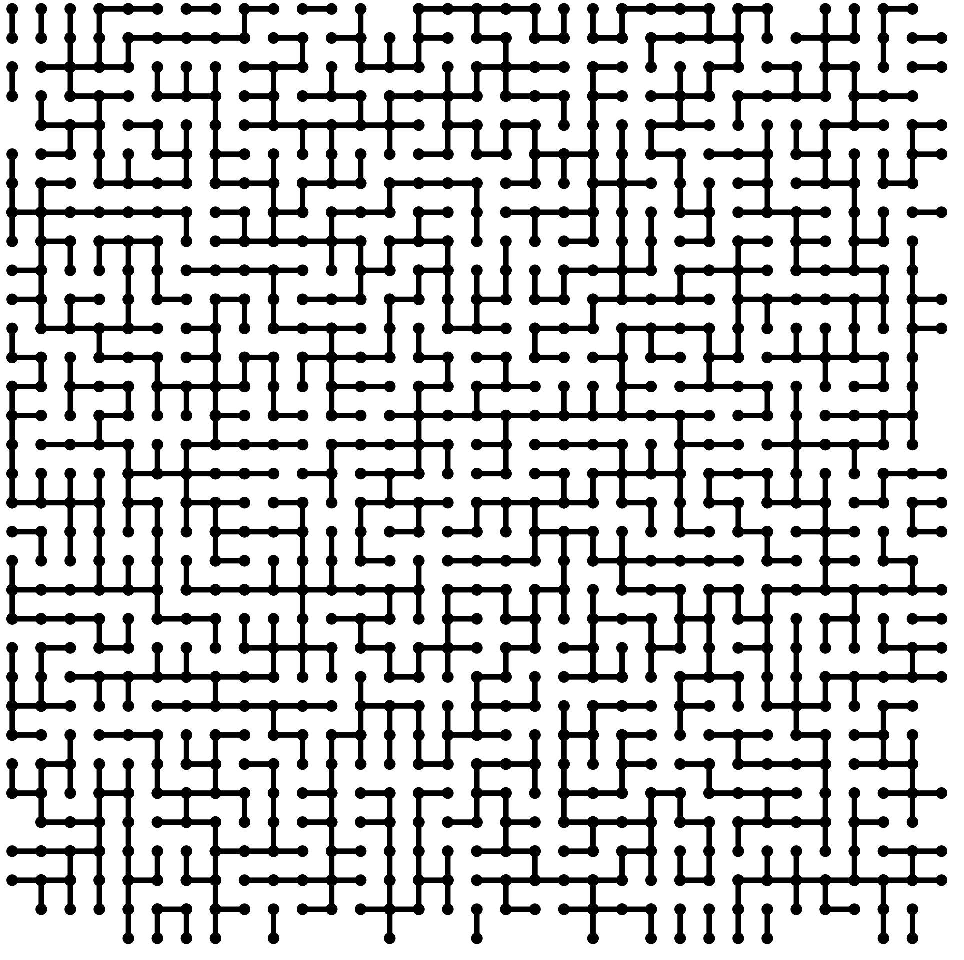


WUSF

Pemantle (1991) showed that these weak limits of the uniform spanning tree measures always exist. These limits are now called the **free uniform spanning forest** on G and the **wired uniform spanning forest**. They are different, e.g., when G is itself a regular tree of degree at least 3.







(David Wilson)

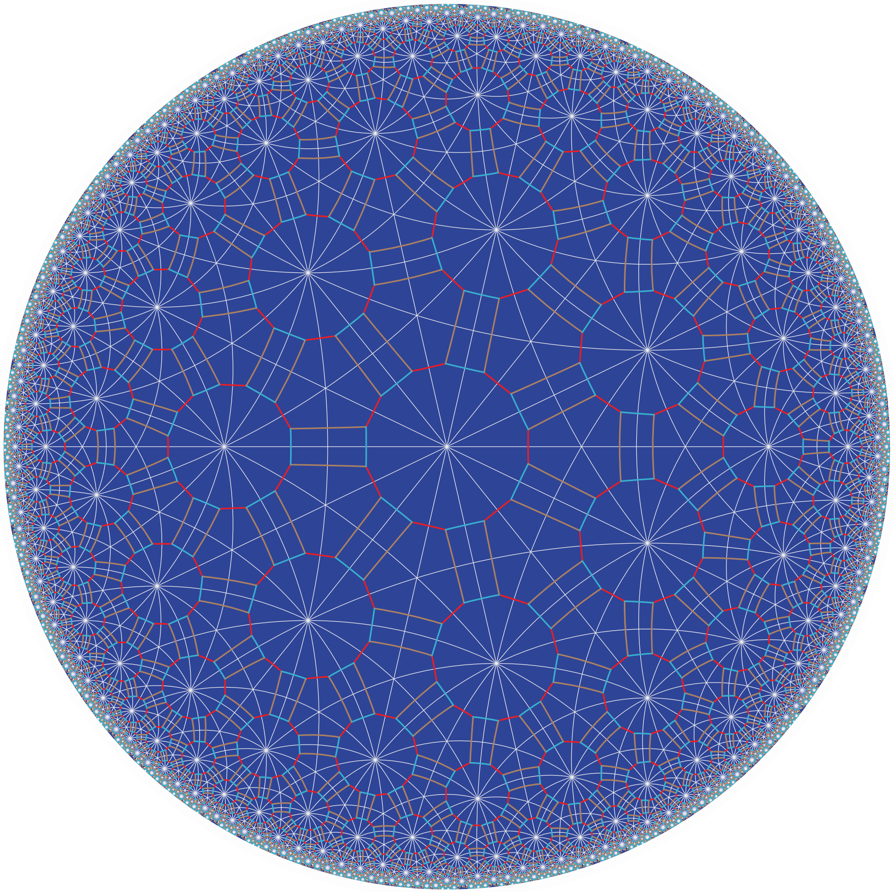
Uniform Spanning Forests on $\mathbb{Z}^d$

Pemantle (1991) discovered the following interesting properties, among others:

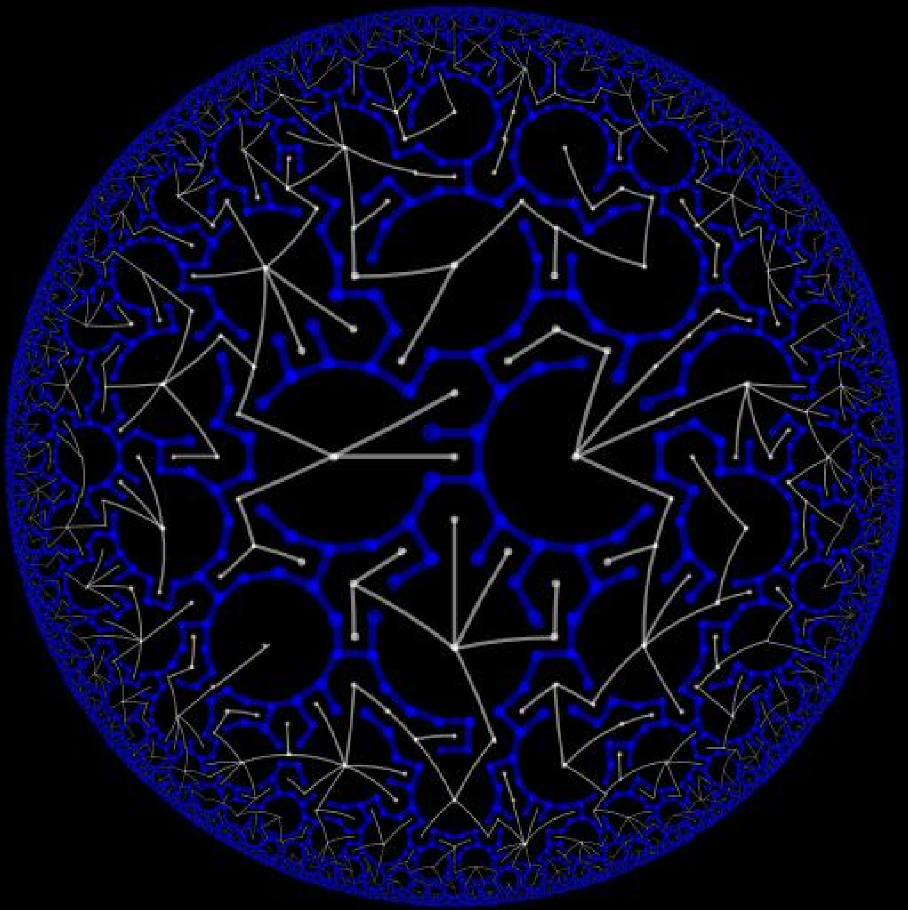
- The free and the wired uniform spanning forest measures are the same on all euclidean lattices $\mathbb{Z}^d$.

- Amazingly, on $\mathbb{Z}^d$, the uniform spanning forest is a single tree a.s. if $d \le 4$; but when $d \ge 5$, there are infinitely many trees a.s.

- If $2 \le d \le 4$, then the uniform spanning tree on $\mathbb{Z}^d$ has a single end a.s.; when $d \ge 5$, each of the infinitely many trees a.s. has at most two ends. Benjamini, Lyons, Peres, and Schramm (2001) showed that each tree has only one end a.s.



A countable group Γ with a finite generating set S gives a Cayley graph G with edges $[\gamma, \gamma s]$ for every $\gamma \in \Gamma$ and $s \in S$.



Suppose that a countable group Γ has a finite generating set S, giving a Cayley graph G with edges $[\gamma, \gamma s]$ for every $\gamma \in \Gamma$ and $s \in S$. Let ∂F denote the vertices outside F that are adjacent to F. We say that Γ is **amenable** if it has a **Følner exhaustion**, i.e., an increasing sequence of finite subsets V_n whose union is Γ such that $\lim_{n\to\infty} |\partial \mathsf{V}_n|/|\mathsf{V}_n| = 0$.

Let G be an amenable infinite Cayley graph with Følner exhaustion $\langle \mathsf{V}_n \rangle$. Let $\mathfrak{F}$ be any deterministic spanning forest all of whose trees are infinite. If k_n denotes the number of trees of $\mathfrak{F} \cap G_n$, then $k_n = o(|\mathsf{V}_n|)$, where G_n is the subgraph of G induced by V_n. Thus, the average degree of vertices is 2:

$$\lim_{n\to\infty} \frac{1}{|\mathsf{V}_n|} \sum_{v \in \mathsf{V}_n} \deg_{\mathfrak{F}}(v) = 2\,.$$

In particular, if $\mathfrak{F}$ is random with an invariant law, such as WUSF or FUSF, then $\mathbf{E}[\deg_{\mathfrak{F}}(v)] = 2$. Because $\mathsf{WUSF} \preccurlyeq \mathsf{FUSF}$, it follows that $\mathsf{WUSF} = \mathsf{FUSF}$ on an amenable Cayley graph.

PROPOSITION. *In every Cayley graph of a group Γ, we have*

$$\mathbf{E}_{\mathsf{WUSF}}[\deg_{\mathfrak{F}} o] = 2 \qquad \textit{(BLPS, 2001)}$$

and

$$\mathbf{E}_{\mathsf{FUSF}}[\deg_{\mathfrak{F}} o] = 2\beta_1(\Gamma) + 2\,. \qquad \textit{(Lyons, 2003)}$$

- $\beta_1(\Gamma) = 0$ if Γ is finite or amenable
- $\beta_1(\Gamma_1 * \Gamma_2) = \beta_1(\Gamma_1) + \beta_1(\Gamma_2) + 1 - \frac{1}{|\Gamma_1|} - \frac{1}{|\Gamma_2|}$
- $\beta_1(\Gamma_1 *_{\Gamma_3} \Gamma_2) = \beta_1(\Gamma_1) + \beta_1(\Gamma_2)$ if Γ_3 is amenable and infinite
- $\beta_1(\Gamma_2) = [\Gamma_1 : \Gamma_2]\beta_1(\Gamma_1)$ if Γ_2 has finite index in Γ_1
- $\beta_1(\Gamma) = 2g - 2$ if Γ is the fundamental group of an orientable surface of genus g
- $\beta_1(\Gamma) = s - 2$ if Γ is torsion free and can be presented with $s \geq 2$ generators and 1 non-trivial relation

If a Cayley graph of Γ has exponential growth, then so does every Cayley graph of Γ. In 1981, Gromov asked whether it must have uniformly exponential growth. Several classes of groups were eventually shown to have uniformly exponential growth, but finally in 2004, J.S. Wilson gave a counter-example.

We'll give another class of groups with uniformly exponential growth and even uniformly positive expansion.

THEOREM (LYONS, PICHOT, AND VASSOUT, 2008). *Let G be a Cayley graph of a finitely generated infinite group Γ with respect to a finite generating set S. For every finite $K \subset \Gamma$, we have*

$$\frac{|\partial K|}{|K|} > 2\beta_1(\Gamma)\,.$$

In particular, this proves that finitely generated groups Γ with $\beta_1(\Gamma) > 0$ have uniform exponential growth. In fact, it shows uniform successive growth of balls, i.e., if

$$\bar{S} := \{\text{identity}\} \cup S \cup S^{-1}\,,$$

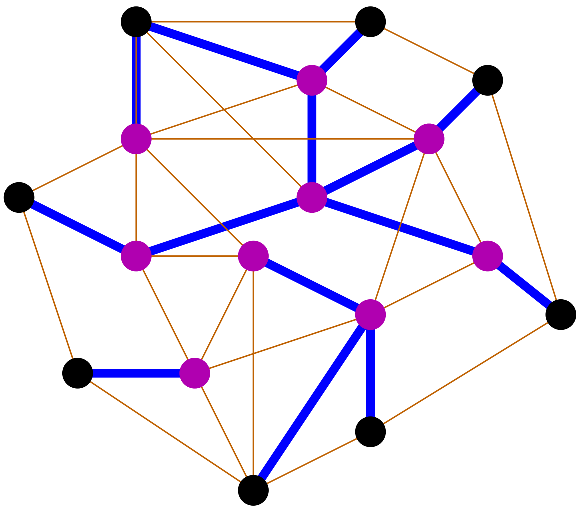
then

$$|\bar{S}^{n+1}|/|\bar{S}^{n}| > 1 + 2\beta_1(\Gamma)\,,$$

so

$$|\bar{S}^{n}| > [1 + 2\beta_1(\Gamma)]^{n}\,.$$

We prove $\frac{|\partial K|}{|K|} > 2\beta_1(\Gamma)$ from $\mathbf{E}_{\mathsf{FUSF}}[\deg_{\mathfrak{F}} o] = 2\beta_1(\Gamma) + 2$.



Proof. Let $\mathfrak{F} \sim \mathsf{FUSF}$. Let $\mathfrak{F}'$ be the part of $\mathfrak{F}$ that touches K. Let $L := \mathsf{V}(\mathfrak{F}') \setminus K$. Since $\mathfrak{F}'$ is a forest,

$$\begin{aligned}\sum_{x \in K} \deg_{\mathfrak{F}} x &\le \sum_{x \in K \cup L} \deg_{\mathfrak{F}'} x - |L| \\ &= 2|\mathsf{E}(\mathfrak{F}')| - |L| \\ &< 2|\mathsf{V}(\mathfrak{F}')| - |L| \\ &= 2|K| + |L| \\ &\le 2|K| + |\partial K| \,.\end{aligned}$$

Take the expectation, use the formula, and divide by $|K|$ to get the result. ∎

If E is finite and $H \subseteq \ell^2(E)$ is a subspace, it defines the determinantal measure

$$\forall T \subseteq E \text{ with } |T| = \dim H \qquad \mathbf{P}^H(T) := \det[P_H]_{T,T}\,,$$

where the subscript T, T indicates the submatrix whose rows and columns belong to T. This representation has a useful extension, namely,

$$\forall D \subseteq E \qquad \mathbf{P}^H[D \subseteq T] = \det[P_H]_{D,D}\,.$$

In case E is infinite and H is a closed subspace of $\ell^2(E)$, the determinantal probability measure $\mathbf{P}^H$ is defined via the requirement that this equation hold for all finite $D \subset E$.

Theorem (Lyons, 2003). *Let E be finite or infinite and let $H \subseteq H'$ be closed subspaces of $\ell^2(E)$. Then $\mathbf{P}^H \preccurlyeq \mathbf{P}^{H'}$, with equality iff $H = H'$.*

This means that there is a probability measure on the set $\{(T, T')\,;\ T \subseteq T'\}$ that projects in the first coordinate to $\mathbf{P}^H$ and in the second to $\mathbf{P}^{H'}$.

Trees, Forests, and Determinants

Let $G = (\mathsf{V}, \mathsf{E})$ be a *finite* graph. Choose one orientation for each edge $e \in \mathsf{E}$. Let $\bigstar = B^1(G)$ denote the subspace in $\ell^2(\mathsf{E})$ spanned by the stars (coboundaries) and let $\diamondsuit = Z_1(G)$ denote the subspace spanned by the cycles. Then $\ell^2(\mathsf{E}) = \bigstar \oplus \diamondsuit$.

For a finite graph, Burton and Pemantle (1993) showed that the uniform spanning tree is the determinantal measure corresponding to orthogonal projection on $\bigstar = \diamondsuit^\perp$. (Precursors due to Kirchhoff (1847) and Brooks, Smith, Stone, and Tutte (1940).)

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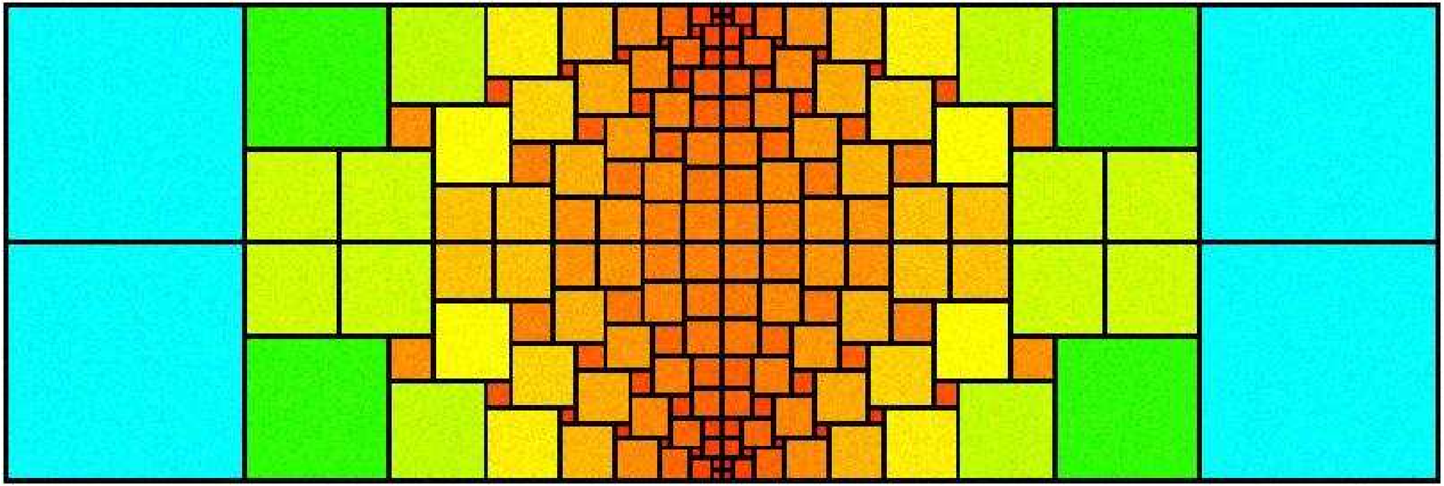

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For an *infinite* graph, let $\bigstar := \bar{B}_c^1(G)$ be the closure in $\ell^2(\mathsf{E})$ of the span of the stars.

For an infinite graph, Benjamini, Lyons, Peres, and Schramm (2001) showed that WUSF is the determinantal measure corresponding to orthogonal projection on $\bigstar$, while FUSF is the determinantal measure corresponding to $\diamondsuit^\perp$.

Thus, $\mathsf{WUSF} \preccurlyeq \mathsf{FUSF}$, with equality iff $\bigstar = \diamondsuit^\perp$.

von Neumann dimension

If $H \subseteq \ell^2(\Gamma)$ is invariant under Γ, then $\dim_\Gamma H$ is a notion of dimension of H per element of Γ: If Γ is finite, then it is just $(\dim H)/|\Gamma| = (\mathrm{tr} P_H)/|\Gamma|$. In general, it is the common diagonal element of the matrix of P_H. More generally, if $H \subseteq \ell^2(\Gamma)^n$ is Γ-invariant, then $\dim_\Gamma H$ is the trace of the common diagonal $n \times n$ block element of the matrix of P_H.

Example: Let $\Gamma := \mathbb{Z}$, so that $\ell^2(\mathbb{Z}) \cong L^2[0,1]$ and H becomes $L^2(A)$ for $A \subseteq [0,1]$. Then $\dim_\mathbb{Z} H = |A|$ since $P_{L^2(A)} f = f\mathbf{1}_A$, so $\dim_\mathbb{Z} H = \int_0^1 (\mathbf{1}\mathbf{1}_A)\mathbf{1} = |A|$.

When $H \subseteq \ell^2(\Gamma)^n$ is Γ-invariant, the probability measure $\mathbf{P}^H$ on subsets of Γ^n is Γ-invariant.

PROPOSITION. *Let G be the Cayley graph of a group Γ with respect to a finite generating set, S. Let o be a vertex of G. Let H be a Γ-invariant closed subspace of $\ell^2(G)$ and $\mathfrak{F} \sim \mathbf{P}^H$. Then*

$$\mathbf{E}^H[\deg_{\mathfrak{F}} o] = 2 \dim_\Gamma H \,.$$

Thus,

$$\mathbf{E}_{\mathsf{FUSF}}[\deg_{\mathfrak{F}} o] = 2 \dim_\Gamma \diamondsuit^\perp$$

and

$$\mathbf{E}_{\mathsf{WUSF}}[\deg_{\mathfrak{F}} o] = 2 \dim_\Gamma \bigstar = 2 \,.$$

We have

$$\beta_1(\Gamma) := \dim_\Gamma \diamondsuit^\perp - \dim_\Gamma \bigstar$$

(ℓ^2-cocycles modulo the closure of the ℓ^2-coboundaries), so

$$\mathbf{E}_{\mathsf{FUSF}}[\deg_{\mathfrak{F}} o] = 2\beta_1(\Gamma) + 2 \,.$$

Proof. Let the standard basis elements of $\ell^2(\Gamma)^{|S|} = \ell^2(\Gamma \times S)$ be $\{\mathbf{1}_{(\gamma,s)}\,;\ \gamma \in \Gamma, s \in S\}$. For simplicity, assume that S contains none of its inverses. Identify E with $\Gamma \times S$ via the map $\langle \gamma, \gamma s\rangle \mapsto (\gamma, s)$. Then H becomes identified with a subspace H_S that is Γ-invariant. Write Q for the orthogonal projection of $\ell^2(\Gamma \times S)$ onto H_S. We may choose o to be the identity of Γ. By Γ-invariance of H,

$$\mathbf{P}^H\big[[s^{-1}, o] \in \mathfrak{F}\big] = \mathbf{P}^H\big[[o, s] \in \mathfrak{F}\big]\,.$$

Therefore,

$$\begin{aligned}
\mathbf{E}^H[\deg_{\mathfrak{F}} o] &= \sum_{s \in S} \mathbf{P}^H\big[[o, s] \in \mathfrak{F}\big] + \sum_{s \in S} \mathbf{P}^H\big[[s^{-1}, o] \in \mathfrak{F}\big] \\
&= 2 \sum_{s \in S} (P_H \mathbf{1}_{[o,s]},\, \mathbf{1}_{[o,s]}) = 2 \sum_{s \in S} (Q \mathbf{1}_{(o,s)},\, \mathbf{1}_{(o,s)}) \\
&= 2\, \dim_\Gamma H_S = 2\, \dim_\Gamma H\,.
\end{aligned}$$

∎

Analogy to Percolation

There is a suggestive analogy to phase transitions in Bernoulli percolation theory. In that theory, given a connected graph G, one considers for $0 < p < 1$ the random subgraph left after deletion of each edge independently with probability $1 - p$. A **cluster** is a connected component of the remaining graph. In the case of Cayley graphs, there are two numbers $p_c, p_u \in [0, 1]$ such that if $0 < p < p_c$, then there are no infinite clusters a.s.; if $p_c < p < p_u$, then there are infinitely many infinite clusters a.s.; and if $p_u < p < 1$, then there is exactly 1 infinite cluster a.s. (Häggström and Peres, 1999).

PROPOSITION. *Let G be a Cayley graph of an infinite group Γ and H be a Γ-invariant closed subspace of $\ell^2(\mathsf{E})$.*

(i) *If $H \subsetneq \bigstar$, then $\mathbf{P}^H$-a.s. infinitely many components of $\mathfrak{F}$ are finite.*

(ii) *If $\bigstar \subseteq H \subsetneq \diamondsuit^\perp$, then $\mathbf{P}^H$-a.s. there are infinitely many infinite components of $\mathfrak{F}$ and no finite components.*

I believe that more is true, namely, that if $H \subsetneq \bigstar$, then $\mathbf{P}^H$-a.s. all components are finite. However, there is no part (iii) in general, i.e., it is *not* true that for every Γ-invariant $H \supsetneq \diamondsuit^\perp$, we have $\mathbf{P}^H$-a.s. there is a unique infinite component, i.e., $\mathbf{P}^H$-a.s. $\mathfrak{F}$ is connected.

Nevertheless, if for every $\epsilon > 0$ there were *some* Γ-invariant $H \supset \diamondsuit^\perp$ with the two properties that $\dim_\Gamma H < \dim_\Gamma \diamondsuit^\perp + \epsilon$ and that $\mathbf{P}^H$-almost every sample is connected, then it would follow that $\beta_1(\Gamma) + 1$ equals the cost of Γ, which would answer an important question of Gaboriau (2002).

An analogous result *is* known for the free minimal spanning forest (Lyons, Peres, and Schramm, 2006). The first property is not hard to ensure, i.e., that for every $\epsilon > 0$ there is some Γ-invariant $H \supset \diamondsuit^\perp$ with $\dim_\Gamma H < \dim_\Gamma \diamondsuit^\perp + \epsilon$.

If $\mathsf{FUSF} \preccurlyeq \mathbf{P}$ with $\mathbf{P}$ invariant and finitely dependent, must $\mathbf{P}$ be carried by connected subgraphs? This would suffice for finitely presented groups (Gaboriau-L.).