Diagram groups and directed 2-complexes: homotopy and homology

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Abstract

We show that diagram groups can be viewed as fundamental groups of spaces of positive paths on directed 2-complexes (these spaces of paths turn out to be classifying spaces). Thus diagram groups are analogs of second homotopy groups, although diagram groups are as a rule non-Abelian. Part of the paper is a review of the previous results from this point of view. In particular, we show that the so called rigidity of the R. Thompson's group F and some other groups is similar to the flat torus theorem. We find several finitely presented diagram groups (even of type $\mathcal{F}_\infty$) each of which contains all countable diagram groups. We show how to compute minimal presentations and homology groups of a large class of diagram groups. We show that the Poincaré series of these groups are rational functions. We prove that all integer homology groups of all diagram groups are free Abelian. We also show that several group theoretic operations on the class of diagram groups correspond to natural operations on directed 2-complexes. For instance, we prove that the class of diagram groups is closed under countable direct products.

Contents

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1 Introduction

The first definition of diagram groups was given by Meakin and Sapir in terms of string rewriting systems (semigroup presentations). Some results about diagram groups were obtained by Meakin's student Vesna Kilibarda (see [21, 22]). Further results about diagram groups have been obtained by the authors of this paper [16, 17, 18], D. Farley [14], and B. Wiest [32].

The definition of diagram groups in terms of string rewriting systems does not reflect the geometry of diagram groups and geometrical nature of the constructions that can be applied to diagram groups. So in this paper we introduce a more geometric definition of diagram groups in terms of directed 2-complexes.

A directed 2-complex (see [31, 26]) is a directed graph equipped with 2-cells each of which is bounded by two directed paths (the top path and the bottom path). With any directed 2-complex one can associate the set of (directed) homotopies or 2-paths which is defined similar to the set of combinatorial homotopies between 1-paths in ordinary combinatorial 2-complexes. Equivalence classes of 2-paths form a groupoid with respect to the natural concatenation of homotopies. The local groups of that groupoid are the diagram groups of the directed 2-complex. Thus, from this point of view, the diagram groups are "directed" analogs of the second homotopy groups.

The new point of view gave us an opportunity to revisit some earlier results about diagram groups. We show that (a multi-dimensional version of) the Squier complex of a semigroup presentation has a natural realization as the space of positive paths in a directed 2-complex. We also show that several facts about diagram groups proved earlier have a natural topological interpretation in terms of directed 2-complexes. So one can consider this paper as, in part, a "revisionistic" survey of our previous work.

The paper also contains completely new results. In particular, we find several diagram groups of type $\mathcal{F}_\infty$ each of which contains all countable diagram groups. One of them has only 3 generators and 6 defining relations. Recall that a group G is said to be of type $\mathcal{F}_\infty$ if there is some $K(G,1)$ complex having a finite n-skeleton in each dimension n.

We study complete directed 2-complexes (they are analogs of complete string rewriting systems). We show how to construct a minimal $K(G,1)$ CW complex (with respect to the number of cells in each dimension) for a diagram group G over a complete directed 2-complex. We compute integer homology of such groups, and show that in the case when the groups are of type $\mathcal{F}_\infty$ (that happens very often), the Poincaré series are rational. We answer Pride's question by showing that the integer homology groups of arbitrary diagram groups are free Abelian. We also study the cohomological dimension of diagram groups of complete directed 2-complexes. In particular, we show that the cohomological dimension of a group in that class is $\geq n$ if and only if the group contains a copy of $\mathbb{Z}^n$ (for any natural number n).

It was shown by Farley [14] that diagram groups of finite semigroup presentations act freely cellularly by isometries on CAT(0) cubical complexes. One of the important results in the theory of CAT(0) groups is the flat torus theorem that shows a rigid connection between a group acting "nicely" on a CAT(0) space, and a geometric property of the space. The algebraic property is "to contain a copy of $\mathbb{Z}^n$", and the geometric property is "to contain a $\mathbb{Z}^n$-invariant copy of $\mathbb{R}^n$". We show that similar rigid connection exists (in our situation) between, say, the R. Thompson group F and the universal cover of the space of positive paths of the Dunce hat.

We show that several group theoretic operations on diagram groups have natural interpretations in terms of directed 2-complexes. In particular, we show that the class of diagram groups is closed under arbitrary (countable) direct products.

The results of this paper are used in our next paper [19] to show that all diagram groups are totally orderable.

2 Combinatorial definition

We start by giving a precise definition of directed 2-complexes. Our definition differs insignificantly from the original definition in [31] and is close to the definition of [26].

Definition 2.1. For every directed graph Γ let $\mathbf{P}$ be the set of all (directed) paths in Γ, including the empty paths. A *directed 2-complex* is a directed graph Γ equipped with a set $\mathbf{F}$ (called the *set of 2-cells*), and three maps $\lceil\cdot\rceil\colon \mathbf{F}\to\mathbf{P}$, $\lfloor\cdot\rfloor\colon\mathbf{F}\to\mathbf{P}$, and ${}^{-1}\colon\mathbf{F}\to\mathbf{F}$ called *top*, *bottom*, and *inverse* such that

- for every $f\in\mathbf{F}$ the paths $\lceil f\rceil$ and $\lfloor f\rfloor$ are non-empty and have common initial vertices and common terminal vertices,
- ${}^{-1}$ is an involution without fixed points, and $\lceil f^{-1}\rceil = \lfloor f\rfloor$, $\lfloor f^{-1}\rfloor = \lceil f\rceil$ for every $f\in\mathbf{F}$.

We shall often need an orientation on $\mathbf{F}$, that is, a subset $\mathbf{F}^+\subseteq\mathbf{F}$ of *positive* 2-cells, such that $\mathbf{F}$ is the disjoint union of $\mathbf{F}^+$ and the set $\mathbf{F}^- = (\mathbf{F}^+)^{-1}$ (the latter is called the set of *negative* 2-cells).

If $\mathcal{K}$ is a directed 2-complex, then paths on $\mathcal{K}$ will be called 1-*paths* (we are going to have 2-paths later). The initial and terminal vertex of a 1-path p will be denoted by $\iota(p)$ and $\tau(p)$ respectively. For every 2-cell $f\in\mathbf{F}$ the vertices $\iota(\lceil f\rceil)=\iota(\lfloor f\rfloor)$ and $\tau(\lceil f\rceil)=\tau(\lfloor f\rfloor)$ are denoted $\iota(f)$ and $\tau(f)$, respectively.

A 2-cell $f\in\mathbf{F}$ with top 1-path p and bottom 1-path q will be called a 2-*cell of the form* $p=q$. We shall use a notation $\mathcal{K}=\langle\,\mathbf{E}\mid\lceil f\rceil=\lfloor f\rfloor,\ f\in\mathbf{F}^+\,\rangle$ for a directed 2-complex with one vertex, the set of edges $\mathbf{E}$ and the set of 2-cells $\mathbf{F}$.

For example, the complex $\langle\,x\mid x^2=x\,\rangle$ is the *Dunce hat* obtained by identifying all edges in the triangle (Figure 1) according to their directions. It has one vertex, one edge, and one positive 2-cell. The remarkable feature of the Dunce hat is that the famous R. Thompson's group F is its diagram group (see Section 6 below). (The survey [13] collects some known results about F. See also [9, 5, 6, 16, 17, 4, 15] for other results about this group.)

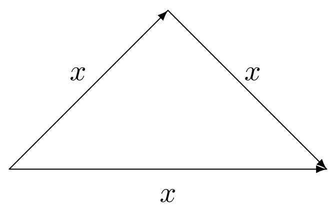

Figure 1.

There exists a natural way to assign a directed 2-complex to every semigroup presentation (to a string rewriting system). It is similar to assigning a 2-complex to any group presentation. If $\mathcal{P}=\langle\,X\mid u_i=v_i\quad(i\in I)\,\rangle$ is a string rewriting system, then the corresponding directed complex $\mathcal{K}_{\mathcal{P}}$ has one vertex, one edge e for each generator from X, and one positive 2-cell for each relation $u_i=v_i$. The top path of this cell is labelled by u_i and the bottom path labelled by v_i.

Every directed 2-complex $\mathcal{K}=\langle\,\mathbf{E}\mid\lceil f\rceil=\lfloor f\rfloor,\ f\in\mathbf{F}^+\,\rangle$ with one vertex can be considered as a rewriting system with the alphabet $\mathbf{E}$ and the set of rewriting rules $\mathbf{F}^+$. The difference between these rewriting systems and string rewriting systems is that there may be several 2-cells in $\mathcal{K}$ with the same top and bottom paths, hence a rewriting rule $p = q$ can repeat many times. We shall observe later that directed 2-complexes with one vertex provide the same class of diagram groups as all directed 2-complexes. But sometimes it is convenient to consider complexes with more than one vertex.

An *atomic 2-path* (*elementary homotopy*) on the directed 2-complex $\mathcal{K}$ consists in replacing $\lceil f \rceil$ by $\lfloor f \rfloor$ in a 1-path. More precisely, it is a triple (p, f, q), where p, q are 1-paths in $\mathcal{K}$, and f is a 2-cell in $\mathcal{K}$ such that $\tau(p) = \iota(f)$, $\tau(f) = \iota(q)$. If δ is the atomic 2-path (p, f, q), then $p\lceil f \rceil q$ is denoted by $\lceil \delta \rceil$, and $p\lfloor f \rfloor q$ is denoted by $\lfloor \delta \rfloor$; these are called the *top* and the *bottom* 1-paths of the 2-path.

For every non-empty 1-path p, we denote by $\varepsilon(p)$ the *trivial* 2-path that consists of p only. Every nontrivial 2-path δ of $\mathcal{K}$ is a sequence of atomic paths $\delta_1, \ldots, \delta_n$, where $\lfloor \delta_i \rfloor = \lceil \delta_{i+1} \rceil$ for every $1 \le i < n$. In this case n is called the *length* of the 2-path δ. The *top* and the *bottom* 1-paths of δ, denoted by $\lceil \delta \rceil$ and $\lfloor \delta \rfloor$, are δ_1 and δ_n, respectively. We say that the 2-path δ *connects* $\lceil \delta \rceil$ with $\lfloor \delta \rfloor$. We also say that $\lceil \delta \rceil$ is (directly) *homotopic* to $\lfloor \delta \rfloor$ in $\mathcal{K}$. We say that δ is *positive* if each δ_i corresponds to a positive 2-cell in $\mathcal{K}$.

If δ', δ'' are 2-paths such that the bottom of δ' coincides with the top of δ'', then one can define a concatenation (product) of them denoted by $\delta' \circ \delta''$ (formally, this is just the sequence δ', δ'').

As in the standard homotopy theory (see, for example, [30]), we need to identify homotopic 2-paths and then define a diagram group $\mathcal{D}(\mathcal{K}, p)$ based at a 1-path p as the group of classes of equivalent 2-paths connecting p with itself. To do this, we choose a computation-friendly way, similar to the one developed by Peiffer, Reidemeister and Whitehead for the second homotopy group of a combinatorial 2-complex, and later simplified by Huebschmann, Sieradski and Fenn (see Bogley and Pride [2]). The idea is to represent the elements of the second homotopy groups in terms of the so called pictures. We are going to use the dual objects called diagrams (for a picture version of this theory see [16]).

With every atomic 2-path $\delta = (p, f, q)$, where $\lceil f \rceil = u$, $\lfloor f \rfloor = v$ we associate the labelled plane graph Δ on Figure 2. An arc labelled by a word w is subdivided into $|w|$ edges[1]. All edges are oriented from the left to the right. The label of each oriented edge of the graph is a symbol from the alphabet $\mathbf{E}$, the set of edges in $\mathcal{K}$. As a plane graph, it has only one bounded face; we label it by the corresponding cell f of $\mathcal{K}$. This plane graph will be called an *atomic diagram over* $\mathcal{K}$. If Δ denotes the diagram, then the leftmost (rightmost) vertex of it is denoted by $\iota(\Delta)$ (resp., $\tau(\Delta)$). The diagram Δ has two distinguished paths from $\iota(\Delta)$ to $\tau(\Delta)$ that correspond to the top and bottom paths of δ. Their labels are puq and pvq, respectively. These are called the top and the bottom paths of Δ denoted by $\lceil \Delta \rceil$ and $\lfloor \Delta \rfloor$.

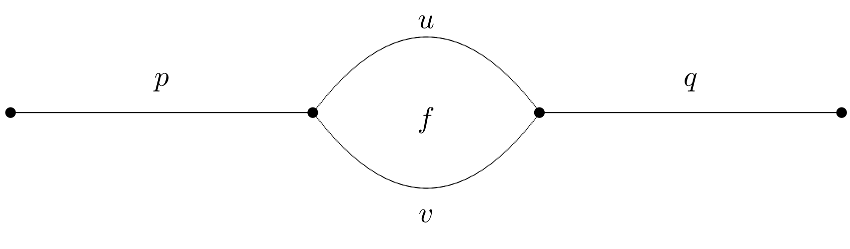

Figure 2.

The diagram corresponding to the trivial 2-path $\varepsilon(p)$ is just an arc labelled by p; it is called a *trivial diagram* and it is denoted by $\varepsilon(p)$ also.

[1]In this paper, we denote the length of a word or a path w by $|w|$.

Let $\delta = \delta_1 \circ \delta_2 \circ \cdots \circ \delta_n$ be a 2-path in $\mathcal{K}$, where $\delta_1, \ldots, \delta_n$ are atomic 2-paths. Let Δ_i be the atomic diagram corresponding to δ_i. Then the bottom path of Δ_i has the same label as the top path of Δ_{i+1}, $1 \le i < n$. Hence we can identify the bottom path of Δ_i with the top path of Δ_{i+1} for all $1 \le i < n$, and obtain a plane graph Δ, which is called the *diagram over* $\mathcal{K}$ *corresponding to the* 2*-path* δ.

It is clear that the above diagram Δ, as a plane graph, has exactly n bounded faces or *cells*. We can regard each of these cells (with its boundary) as a diagram itself and thus apply the functions ι, τ, $\lceil\cdot\rceil$, $\lfloor\cdot\rfloor$ to each of the cells.

Two diagrams are considered *equal* if they are isotopic as plane graphs. The isotopy must preserve vertices and edges, it must also preserve labels of edges and inner labels of cells. Two 2-paths are called $\approx$*-similar* if the corresponding diagrams are isotopic.

For example, let $u_1 = v_1$, $u_2 = v_2$ be two 2-cells $f_1, f_2 \in \mathbf{F}$ (we allow $f_1 = f_2$) and let p, q, r be some 1-paths in $\mathcal{K}$ such that $\tau(p) = \iota(f_1)$, $\tau(f_1) = \iota(q)$, $\tau(q) = \iota(f_2)$, $\tau(f_2) = \iota(r)$. Then the corresponding diagram over $\mathcal{K}$ is on Figure 3 below. In this case we say that the atomic 2-paths (p, f_1, qu_2r) and (pu_1q, f_2, r) are *independent*. It is clear that this diagram corresponds to the 2-path

$$(p, f_1, qu_2r) \circ (pv_1q, f_2, r) \tag{1}$$

as well as the 2-path

$$(pu_1q, f_2, r) \circ (p, f_1, qv_2r). \tag{2}$$

It is easy to show that the relation $\approx$ is the smallest equivalence relation containing pairs of 2-paths of the form (1) and (2) and invariant under concatenation (in [16], we in fact proved that in the case of directed 2-complexes corresponding to semigroup presentations).

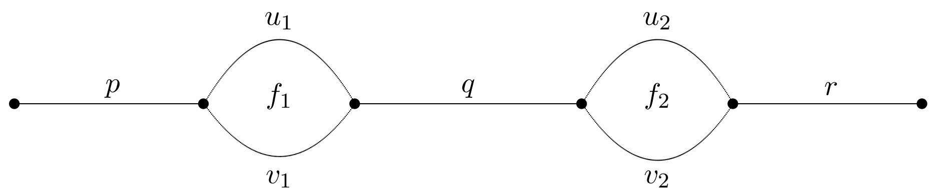

Figure 3.

Concatenation of 2-paths corresponds to the concatenation of diagrams: if the bottom path of Δ_1 and the top path of Δ_2 have the same labels, we can identify them and obtain a new diagram $\Delta_1 \circ \Delta_2$.

Note that for any 2-path $\delta = (p, f, q)$ in $\mathcal{K}$ one can naturally define its *inverse* 2-path $\delta^{-1} = (p, f^{-1}, q)$. The same can be done for diagrams. The inverse diagram Δ^{-1} for Δ is just the mirror image of Δ with respect to a horizontal line, where the labels of cells are replaced by their inverses.

All classes of $\approx$-similar 2-paths in $\mathcal{K}$ form a partial monoid under concatenation, empty paths are the identities. To obtain a groupoid we need to identify $\delta \circ \delta^{-1}$ with the empty 2-path $\varepsilon(\lceil\delta\rceil)$ for every 2-path δ. The corresponding quotient of the partial monoid is called the *diagram groupoid* of $\mathcal{K}$, denoted by $\mathcal{D}(\mathcal{K})$. Two 2-paths are called *equivalent* if they define the same element in the diagram groupoid. The local groups of this groupoid are the equivalence classes of 2-paths that connect a given 1-path p with itself. These groups are called the *diagram groups* of the directed 2-complex $\mathcal{K}$ with *base* p. We denote them by $\mathcal{D}(K, p)$. Notice that if p is empty then $\mathcal{D}(\mathcal{K}, p)$ is trivial by definition. In this paper, we shall usually ignore these diagram groups.

Remark 2.2. Notice first that the diagram groups of a directed 2-complex do not depend on the orientation on the set of 2-cells of that complex. Notice also that if a directed 2-complex $\mathcal{K}'$ is obtained from $\mathcal{K}$ by identifying vertices, then the diagram groupoid of $\mathcal{K}'$ may differ from the diagram groupoid of $\mathcal{K}$ because the set of 1-paths may increase, but the diagram groups of $\mathcal{K}$ will be diagram groups of $\mathcal{K}'$ as well.

One can easily check that if $\mathcal{K} = \mathcal{K}_{\mathcal{P}}$ for some semigroup presentation $\mathcal{P}$ and w is a word over X (that is, the corresponding path in $\mathcal{K}_{\mathcal{P}}$), then the diagram group $\mathcal{D}(\mathcal{K}, w)$ we just defined coincides with the diagram group $D(\mathcal{P}, w)$ over $\mathcal{P}$ defined in [16]. Clearly, if $\mathcal{K} = \mathcal{K}_{\mathcal{P}}$ then 2-paths are just the derivations over the semigroup presentation $\mathcal{P}$.

It is convenient to define diagrams over a directed 2-complex $\mathcal{K}$ in an "abstract" way, without referring to 2-paths of $\mathcal{K}$. Such a definition was given by Kashintsev [20] and Remmers [27] in the case of semigroup presentations. Here we basically repeat their definition and result.

Definition 2.3. A diagram over $\mathcal{K} = \langle \mathbf{E} \mid \lceil f \rceil = \lfloor f \rfloor,\ f \in \mathbf{F}^+ \rangle$ is a finite plane directed and connected graph Δ, where every edge is labelled by an element from $\mathbf{E}$, and every bounded face is labelled by an element of $\mathbf{F}$ such that:

- Δ has exactly one vertex-source $\imath$ (which has no incoming edges) and exactly one vertex-sink τ (which has no outgoing edges);
- every 1-path in Δ is simple;
- each face of Δ labelled by $f \in \mathbf{F}$ is bounded by two 1-paths u and v such that the label of u is $\lceil f \rceil$, the label of v is $\lfloor f \rfloor$, and the loop uv^{-1} on the plane goes around the face in the clockwise direction.

It is easy to see [16] that every plane graph satisfying the conditions of Definition 2.3 is situated between two positive paths connecting $\imath$ and τ. These paths are $\lceil \Delta \rceil$ and $\lfloor \Delta \rfloor$.

We say that a diagram Δ over a directed 2-complex $\mathcal{K}$ is a (u, v)-*diagram* whenever u is the top label and v is the bottom label of Δ. If u and v are the same, then the diagram is called *spherical* (with base $u = v$).

The following lemma (see [16, Lemma 3.5]) shows that diagrams over $\mathcal{K}$ in the sense of Definition 2.3 are exactly diagrams that correspond to 2-paths in $\mathcal{K}$. We will often use this fact without reference.

Lemma 2.4. *Let $\mathcal{K}$ be a directed 2-complex. Then any 1-paths u, v are homotopic in $\mathcal{K}$ if and only if there exists a (u, v)-diagram over $\mathcal{K}$ (in the sense of Definition 2.3).*

Diagrams over $\mathcal{K}$ corresponding to equivalent 2-paths are also called *equivalent*. The equivalence relation on the set of diagrams and on the set of 2-paths of $\mathcal{K}$ can be defined very easily as follows. We say that two cells π_1 and π_2 in a diagram Δ over $\mathcal{K}$ form a *dipole* if $\lfloor \pi_1 \rfloor$ coincides with $\lceil \pi_2 \rceil$ and the labels of the cells π_1 and π_2 are mutually inverse. Clearly, if π_1 and π_2 form a dipole, then one can remove the two cells from the diagram and identify $\lceil \pi_1 \rceil$ with $\lfloor \pi_2 \rfloor$. As in [16], it is easy to prove that if δ is a 2-path corresponding to Δ then the resulting diagram Δ' corresponds to a 2-path δ', which is equivalent to δ. We call a diagram *reduced* if it does not contain dipoles. A 2-path δ in $\mathcal{K}$ is called *reduced* if the corresponding diagram is reduced. The following is an analog of Kilibarda's lemma. The proof coincides with the proof of [16, Theorem 3.17] and we omit it here.

Theorem 2.5. *Every equivalence class of diagrams over a directed* 2*-complex* $\mathcal{K}$ *contains exactly one reduced diagram. Every* 2*-path in* $\mathcal{K}$ *is equivalent to a reduced* 2*-path, every two equivalent reduced* 2*-paths have equal diagrams and so they contain the same number of atomic factors.*

Thus one can define the diagram groupoid $\mathcal{D}(\mathcal{K})$ of a directed 2-complex $\mathcal{K}$ as the set of reduced diagrams over $\mathcal{K}$ with operation "concatenation + reduction" (i. e., the product of two reduced diagrams Δ and Δ' is the result of removing dipoles from $\Delta \circ \Delta'$).

The diagram groupoid $\mathcal{D}(\mathcal{K})$ has another natural operation, *addition*: if Δ' and Δ'' are diagrams over $\mathcal{K}$ and $\tau(\lceil\Delta'\rceil) = \iota(\lceil\Delta''\rceil)$ in $\mathcal{K}$ then one can identify $\tau(\Delta')$ with $\iota(\Delta'')$ to obtain the new diagram denoted by $\Delta' + \Delta''$ and called the *sum* of Δ' and Δ''. If $\mathcal{K}$ has only one vertex, then this operation is everywhere defined.

Figure 4 below illustrates the concepts of the concatenation of diagrams and the sum of them.

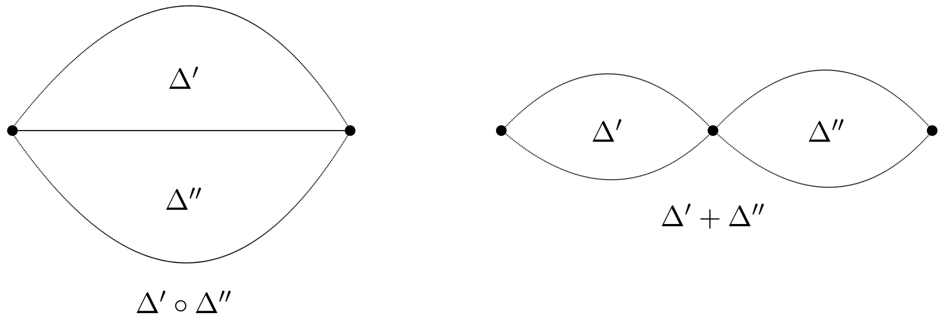

Figure 4.

3 Topological definition

We have seen that diagram groups are directed analogs of the second homotopy groups. Recall that one can define the second homotopy groups of a topological space as the fundamental group of a space of paths. On the other hand, in the case of semigroup presentations, the diagram groups can be defined as fundamental groups of the so called 2-dimensional Squier complexes associated with the presentations (see [16]).

In this section, we show that the Squier complex of a directed 2-complex $\mathcal{K}$ can be considered as the space of certain positive paths in $\mathcal{K}$.

First we define a multidimensional version of the Squier complex $\mathrm{Sq}(\mathcal{K})$ from [16] as a semi-cubical complex. Recall [29, 10] that a *semi-cubical complex* is a family of sets $\{M_n, n \geq 0\}$ (elements of M_n are called n-cubes) with face maps $\lambda_i^k \colon M_n \to M_{n-1}$ $(1 \leq i \leq n,\ k = 0, 1)$ satisfying the semi-cubical relations:

$$\lambda_i^k \lambda_j^{k'} = \lambda_{j-1}^{k'} \lambda_i^k \quad (i < j). \tag{3}$$

A *realization* of a semi-cubical complex $\{M_n, n \geq 0\}$ can be obtained as a factor-space of the disjoint union of Euclidean cubes, one n-cube $c(x)$ for each element $x \in M_n$, $n \geq 0$. The equivalence relation identifies (point-wise) the cube $\lambda_i^k(c(x))$ with the cube $c(\lambda_i^k(x))$ for all i, k, x. Here $\lambda_i^k(I^n)$ is the corresponding $(n-1)$-face of the Euclidean n-cube I^n (i. e., $\lambda_i^k(I^n) = I^{i-1} \times \{k\} \times I^{n-i}$).

Definition 3.1. The semi-cubical complex $\mathrm{Sq}(\mathcal{K})$ is defined as follows. For every $n \geq 0$ let M_n be the set of *thin* diagrams [14] of the form

$$\varepsilon(u_0) + f^{(1)} + \varepsilon(u_1) + \cdots + f^{(n)} + \varepsilon(u_n) \tag{4}$$

where $f^{(i)}$ are negative[2] 2-cells of $\mathcal{K}$ and u_i are 1-paths in $\mathcal{K}$ (Figure 3 thus shows a thin diagram with two cells). The face map λ_i^k takes the thin diagram c of the form (4) to

$$\lceil c \rceil_i = \varepsilon(u_0) + f^{(1)} + \cdots + \lceil f^{(i)} \rceil + \cdots + f^{(n)} + \varepsilon(u_n) \tag{5}$$

if $k = 0$ and

$$\lfloor c \rfloor_i = \varepsilon(u_0) + f^{(1)} + \cdots + \lfloor f^{(i)} \rfloor + \cdots + f^{(n)} + \varepsilon(u_n) \tag{6}$$

if $k = 1$. These faces are called the *top* and the *bottom* ith faces of c, respectively. It is easy to check that the conditions (3) are satisfied, so $\mathrm{Sq}(\mathcal{K})$ is a semi-cubical complex.

Thus the vertices of $\mathrm{Sq}(\mathcal{K})$ are 1-paths of $\mathcal{K}$, the edges correspond to negative atomic 2-paths (u, f, v), 2-cells correspond to pairs of independent atomic 2-paths, etc. For example, the thin diagram $\varepsilon(u) + f + \varepsilon(v) + g + \varepsilon(w)$, where $f, g \in \mathbf{F}^-$ determines a square with contour

$$(u, f, v\lceil g \rceil w) \circ (u\lfloor f \rfloor v, g, w) \circ (u, f^{-1}, v\lfloor g \rfloor w) \circ (u\lceil f \rceil v, g^{-1}, w).$$

It is convenient to enrich the structure of the Squier complex $\mathrm{Sq}(\mathcal{K})$ by introducing inverse edges: $(u, f, v)^{-1} = (u, f^{-1}, v)$. Then the edges (u, f, v) will be called *positive* if f is a positive 2-cell of $\mathcal{K}$, and *negative* if f is negative. As a result, the 1-skeleton of $\mathrm{Sq}(\mathcal{K})$ turns into a graph in the sense of Serre [28], and the 2-skeleton of $\mathrm{Sq}(\mathcal{K})$ coincides with the Squier complex defined in [16] provided $\mathcal{K} = \mathcal{K}_{\mathcal{P}}$ for some $\mathcal{P}$. Hence, in particular, the fundamental groups of $\mathrm{Sq}(\mathcal{K}_{\mathcal{P}})$ coincide with fundamental groups of the Squier complex in [16].

Clearly the complex $\mathrm{Sq}(\mathcal{K})$ is in general disconnected. If p is a 1-path in $\mathcal{K}$, then by $\mathrm{Sq}(\mathcal{K}, p)$ we will denote the connected component of the Squier complex that contains p.

Example 3.2. Figure 5 shows a part of the Squier complex of the Dunce hat on Figure 1. The thick line shows the boundary of one of the 2-cells in this complex.

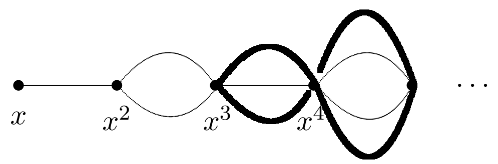

Figure 5.

The following theorem is similar to Kilibarda's statement (see [16, Theorem 6.1]).

Theorem 3.3. *Let $\mathcal{K}$ be a directed 2-complex, p be a 1-path in $\mathcal{K}$. Then the diagram group $\mathcal{D}(\mathcal{K}, p)$ is isomorphic to the fundamental group $\pi_1(\mathrm{Sq}(\mathcal{K}), p)$ of the semi-cubical complex $\mathrm{Sq}(\mathcal{K})$ with the basepoint p.*

The proof of Kilibarda's theorem carries without any essential changes. As an immediate corollary of Theorem 3.3, we obtain the following

[2]Taking negative edges instead of positive simplifies some computations later.

Corollary 3.4. *Let $\mathcal{K}$ be a directed 2-complex, p and q be homotopic 1-paths in $\mathcal{K}$. Then $\mathcal{D}(\mathcal{K}, p)$ is isomorphic to $\mathcal{D}(\mathcal{K}, q)$.*

Proof. Indeed, p and q belong to the same connected component of the Squier complex $\mathrm{Sq}(\mathcal{K})$. □

The diagram groups with different bases that are fundamental groups of different connected components of $\mathrm{Sq}(\mathcal{K})$ can be very different but there exists the following useful relationship between them.

Corollary 3.5. *If $p = p_1p_2$ is a 1-path in $\mathcal{K}$, then $\mathcal{D}(\mathcal{K}, p_1) \times \mathcal{D}(\mathcal{K}, p_2)$ is embedded into $\mathcal{D}(\mathcal{K}, p_1p_2)$.*

Proof. Indeed, the map $(\Delta_1, \Delta_2) \to \Delta_1 + \Delta_2$ from $\mathcal{D}(\mathcal{K}, p_1) \times \mathcal{D}(\mathcal{K}, p_2)$ to $\mathcal{D}(\mathcal{K}, p_1p_2)$ is an injective homomorphism (see [16], Remark 2 after Lemma 8.1). □

Now let us introduce a natural topological realization of the semi-cubical complex $\mathrm{Sq}(\mathcal{K})$.

We expand the set of paths in $\mathcal{K}$ allowing paths that go "inside" 2-cells.

Let $\mathcal{K}$ be a directed 2-complex. Attaching a 2-cell $f \in \mathbf{F}^+$ with $p = \lceil f \rceil$, $q = \lfloor f \rfloor$ can be done as follows. Let $D = [0,1] \times [0,1]$ be a unit square. For any $t \in [0,1]$ we have the path d_t in D defined by $d_t(s) = (s,t) \in D$ $(s \in [0,1])$. We attach this square to $\mathcal{K}$ in such a way that d_0 is identified with p, d_1 is identified with q, all points of the form $(0,t) \in D$ are collapsed to $\iota(p) = \iota(q)$, all points of the form $(1,t)$ are collapsed to $\tau(p) = \tau(q)$ $(t \in [0,1])$.

Now for any $t \in [0,1]$, the image of d_t in $\mathcal{K}$ becomes a path inside the 2-cell f. This path will be denoted by f_t. Clearly, $f_0 = p$, $f_1 = q$. So we have a continuous family of paths $\{f_t\}$ $(t \in [0,1])$ that transforms p into q (see Figure 6).

For any $f \in \mathbf{F}^+$, $t \in [0,1]$ one can also define $(f^{-1})_t = f_{1-t}$. So f_t makes sense for any $f \in \mathbf{F}$.

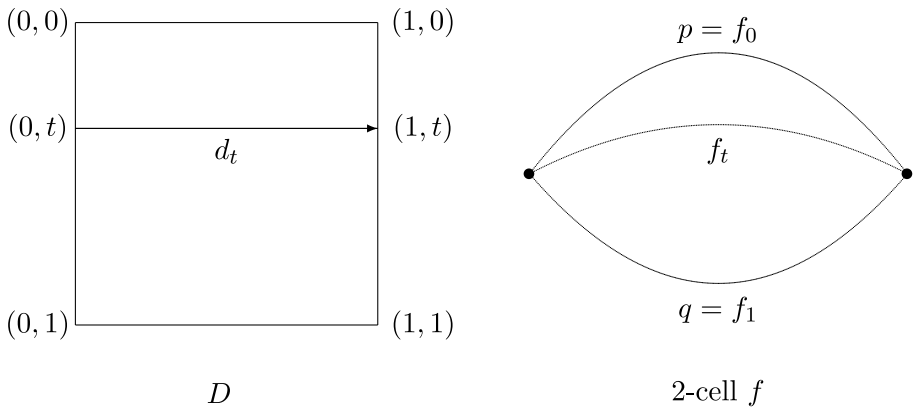

Figure 6.

By definition, a *positive path in a directed 2-complex* $\mathcal{K}$ is a finite product of the form $p_1 \cdots p_n$ $(n \geq 1)$, where each factor p_i $(1 \leq i \leq n)$ is either a 1-path on $\mathcal{K}$, or a path of the form f_t for some $f \in \mathbf{F}$, $t \in [0,1]$. Of course, we assume that the terminal point of p_i coincides with the initial point of p_{i+1} for any $1 \leq i < n$ (see Figure 7). The set of all positive paths in $\mathcal{K}$ defined in this way will be denoted by $\Omega_+(\mathcal{K})$. Note that every 1-path in $\mathcal{K}$ is a positive path in this sense.

One can easily see that $\Omega_+(\mathcal{K})$ can be viewed as a realization of the semi-cubical complex $\mathrm{Sq}(\mathcal{K})$: with every non-empty positive path $p = u_0 f^{(1)}_{t_1} u_1 \cdots f^{(n)}_{t_n} u_n$ we assign the point with coordinates $(t_1, \ldots, t_n)$ in the n-cube $\varepsilon(u_0) + f^{(1)} + \ldots + f^{(n)} + \varepsilon(u_n)$. Empty paths correspond to isolated points in the Squier complex.

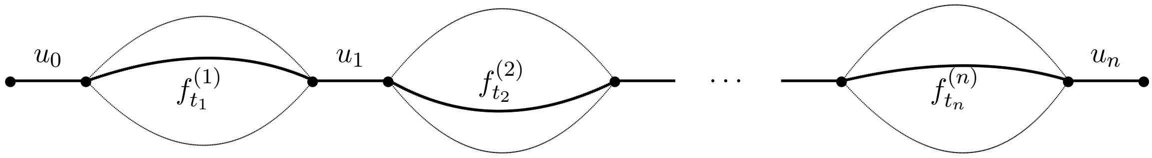

Figure 7.

So we have an equivalent definition of diagram groups of $\mathcal{K}$ as the fundamental groups of the space of positive paths in $\mathcal{K}$. One possible way of generalizing the definition of diagram groups and of defining "continuous versions" of the diagram groups would be to consider a more general spaces than directed 2-complexes, define "positive paths" in a suitable way, and then to consider the fundamental groups of the spaces of positive paths. They may have certain properties in common with diagram groups.

It is useful to have an explicit procedure to establish the correspondence between diagrams and paths in $\Omega_+(\mathcal{K})$. Each atomic diagram has the form $\varepsilon(p) + f + \varepsilon(q)$, where p, q are 1-paths in $\mathcal{K}$ and $f \in \mathbf{F}$ is a 2-cell of $\mathcal{K}$. The family of positive paths $p \cdot f_t \cdot q$ $(t \in [0,1])$ is a path in $\Omega_+(\mathcal{K})$. It continuously deforms $p\lceil f \rceil q$ to $p\lfloor f \rfloor q$ in $\Omega_+(\mathcal{K})$. So any atomic diagram Δ over $\mathcal{K}$ corresponds to a path in $\Omega_+(\mathcal{K})$ that connects $\lceil \Delta \rceil$ and $\lfloor \Delta \rfloor$. Every diagram can be decomposed into a product of atomic diagrams, so every diagram over $\mathcal{K}$ corresponds naturally to a path in $\Omega_+(\mathcal{K})$.

We shall not distinguish between the Squier complex $\mathrm{Sq}(\mathcal{K})$ and its geometric realization. The universal cover $\widetilde{\mathrm{Sq}}(\mathcal{K})$ of the Squier complex $\widetilde{\mathrm{Sq}}(\mathcal{K})$ has been studied in detail by Farley [14]. He proved that each connected component of $\widetilde{\mathrm{Sq}}(\mathcal{K})$ is contractible if $\mathcal{K} = \mathcal{K}_{\mathcal{P}}$ for some $\mathcal{P}$. In fact $\widetilde{\mathrm{Sq}}(\mathcal{K})$ can be described in just the same way as $\mathrm{Sq}(\mathcal{K})$, only the vertices of $\widetilde{\mathrm{Sq}}(\mathcal{K})$ are not 1-paths but diagrams over $\mathcal{K}$. The restriction of the covering map $\widetilde{\mathrm{Sq}}(\mathcal{K}) \to \mathrm{Sq}(\mathcal{K})$ to the vertices is $\lfloor \cdot \rfloor$. Farley's proof also carries without any change to the case of arbitrary directed 2-complexes. This implies the following important result.

Theorem 3.6. *The universal cover* $\widetilde{\mathrm{Sq}}(\mathcal{K}, p)$ *is contractible. Hence* $\mathrm{Sq}(\mathcal{K}, p)$ *is a* $K(G, 1)$ *complex for the diagram group* $G = \mathcal{D}(\mathcal{K}, p)$*, for every directed* 2*-complex* $\mathcal{K}$ *and every* 1*-path* p *in* $\mathcal{K}$*.*

Another important feature of $\widetilde{\mathrm{Sq}}(\mathcal{K}, p)$ is that it is a cubical complex in the sense of [3] and has the CAT(0) property provided, for example, $\mathcal{K}$ is a finite complex (this is essentially proved in [14]). Thus every diagram group of a finite directed 2-complex acts freely and cellularly by isometries on a CAT(0) cubical complex.

4 Theorems about isomorphism. The class of diagram groups

In this section, we show that diagram groups do not change much if we do certain surgeries on directed 2-complexes.

The following useful statement contains a directed 2-complex analog of Tietze transformations for group and semigroup presentations.

Theorem 4.1. *Let* $\mathcal{K}$ *be a directed* 2*-complex.*

1)*. Let* u *be a non-empty* 1*-path in* $\mathcal{K}$*. Let* $\mathcal{K}'$ *be the directed* 2*-complex obtained from* $\mathcal{K}$ *by adding a new edge* e *with* $\iota(e) = \iota(u)$*,* $\tau(e) = \tau(u)$ *and a new* 2*-cell* f *of the form* $u = e$ *(and* *also the inverse* 2*-cell* f^{-1}*). Then for every* 1*-path* w *in* $\mathcal{K}$*, the diagram groups* $\mathcal{D}(\mathcal{K}, w)$ *and* $\mathcal{D}(\mathcal{K}', w)$ *are isomorphic.*

2). *Suppose that* $\mathcal{K}$ *is a union of two directed* 2*-complexes* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *such that all vertices and edges of* $\mathcal{K}$ *are both in* $\mathcal{K}_1$ *and in* $\mathcal{K}_2$*. Suppose that the top path* $\lceil f \rceil$ *(bottom path* $\lfloor f \rfloor$*) of each positive* 2*-cell* f *of* $\mathcal{K}_1$ *is homotopic in* $\mathcal{K}_2$ *to some path* u_f *(resp.,* v_f*). Let us consider the directed complex* $\mathcal{K}'$ *with the same vertices and edges as* $\mathcal{K}$ *and* 2*-cells from* $\mathcal{K}_2$ *together with all positive* 2*-cells* $u_f = v_f$ *for all positive* 2*-cells* f *from* $\mathcal{K}_1$ *(plus the corresponding negative cells* $v_f = u_f$*). Then the diagram groups* $\mathcal{D}(\mathcal{K}, w)$ *and* $\mathcal{D}(\mathcal{K}', w)$ *are isomorphic for every* 1*-path* w *in* $\mathcal{K}$*.*

Proof. 1. Indeed, there exists a natural embedding of $\mathcal{D}(\mathcal{K}, w)$ into $\mathcal{D}(\mathcal{K}', w)$ which maps every reduced (w, w)-diagram over $\mathcal{K}$ to itself. In order to show that this map is surjective, notice that if a (w, w)-diagram Δ over $\mathcal{K}'$ does not contain edges labelled by e then it is a diagram over $\mathcal{K}$. If Δ contains an edge labelled by e then this edge cannot be on $\lceil \Delta \rceil$ or $\lfloor \Delta \rfloor$. Hence this edge is a common edge of the contours of two cells π and π' in Δ. Since $\mathcal{K}'$ has only two 2-cells with e on the boundary (namely, f and f^{-1}), one of the two cells π or π' is labelled by f and another by f^{-1} (the edge labelled by e is the top path of one of these cells and the bottom path of another one). Hence π and π' form a dipole. This implies that every reduced (w, w)-diagram over $\mathcal{K}'$ contains no edges labelled by e, and so it is a diagram over $\mathcal{K}$. Hence by Theorem 2.5 the natural embedding of $\mathcal{D}(\mathcal{K}, w)$ into $\mathcal{D}(\mathcal{K}', w)$ is surjective.

2. By Theorem 2.4, for any 2-cell f of $\mathcal{K}_1$ there exist diagrams Γ_f and Δ_f over $\mathcal{K}_2$ such that the label of $\lceil \Gamma_f \rceil$ is $\lceil f \rceil$, the label of $\lfloor \Gamma_f \rfloor$ is u_f, the label of $\lceil \Delta_f \rceil$ is $\lfloor f \rfloor$, the label of $\lfloor \Delta_f \rfloor$ is v_f.

By D_w (respectively, D'_w) we denote the set of all (w, w)-diagrams over $\mathcal{K}$ (respectively $\mathcal{K}'$). We are going to define two maps $\phi: D_w \to D'_w$, $\psi: D'_w \to D_w$.

Let $\Xi \in D_w$. Let $\mathbf{F}_1^+$ be the set of positive 2-cells in $\mathcal{K}_1$. For every cell π in Ξ with inner label $f \in \mathbf{F}_1^+$, we do the following operation. First we cut π into three parts by connecting the initial vertex of Π with the terminal vertex of π by two simple curves, p_1 and p_2, that have no intersections other than at the endpoints. We enumerate the three parts from top to bottom and assume that p_1 is above p_2. Then we subdivide p_1 into edges and give them labels such that p_1 will have label u_f. Similarly, we turn p_2 into a path labelled by v_f. Now we insert the diagram Γ_f between the top path of π and p_1. Analogously, we insert Δ_f^{-1} between p_2 and the bottom path of π. The space between p_1 and p_2 becomes a cell $u_f = v_f$, which is a cell $\mathcal{K}'$. We can assign the inner label f to it. If the inner label of a cell π of Ξ is $f^{-1} \in \mathbf{F}_1^-$, then we subdivide it in the same way to get the mirror image of the diagram we had for cells with inner label f. (The inner label for the cell in the middle will be f^{-1}.)

Every diagram Ξ over $\mathcal{P}$ now becomes a diagram over $\mathcal{P}'$. We denote it by $\phi(\Xi)$.

The map ψ is defined similarly. Now if we have a diagram Ξ over $\mathcal{K}'$, then we replace each of its cells π of the form $u_f = v_f$ ($f \in \mathbf{F}_1$) by the concatenation of three diagrams. The first of them is Γ_f^{-1}, the third is Δ_f, and the second one is a cell with inner label f. We do similar transformation with cells of the form $v_f = u_f$ whose inner labels are negative.

The result of these replacements will be a diagram over $\mathcal{K}$ denoted by $\psi(\Xi)$.

It follows from our construction that for any diagram Ξ over $\mathcal{K}$, the diagram $\psi(\phi(\Xi))$ over $\mathcal{K}$ is equivalent to Ξ. This is so because after applying ϕ and then ψ to Ξ, we get a diagram with a number of subdiagrams of the form $\Gamma^{\pm 1}\Gamma^{\mp 1}$ or $\Delta^{\pm 1}\Delta^{\mp 1}$. Cancelling all the dipoles, we get the diagram Ξ we had in the beginning. Analogously, for any diagram Ξ over $\mathcal{K}'$, the diagram $\phi(\psi(\Xi))$ over $\mathcal{K}'$ will be equivalent to Ξ. It is also clear that ϕ and ψ preserve the operation of concatenation of diagrams.

This means that maps ϕ, ψ induce homomorphisms of diagram groups $\bar{\phi}: \mathcal{D}(\mathcal{K}, w) \to \mathcal{D}(\mathcal{K}', w)$ and $\bar{\psi}: \mathcal{D}(\mathcal{K}', w) \to \mathcal{D}(\mathcal{K}, w)$. The fact about equivalence of diagrams means that $\bar{\phi}$ and $\bar{\psi}$ are mutually inverse. Thus they are isomorphisms and $\mathcal{D}(\mathcal{K}, w) \cong \mathcal{D}(\mathcal{K}', w)$. □

As an immediate application of Theorem 4.1, we obtain the following statement about subdivisions of directed 2-complexes. Let $\mathcal{K}$ be a directed 2-complex and let f be its 2-cell. Let us add a new edge e to the complex with $\iota(e) = \iota(\lceil f \rceil) = \iota(\lfloor f \rfloor)$ and $\tau(e) = \tau(\lceil f \rceil) = \tau(\lfloor f \rfloor)$, remove the 2-cells $f^{\pm 1}$, and add new 2-cells $f_1^{\pm 1}$, $f_2^{\pm 1}$, where f_1, f_2 have the form $\lceil f \rceil = e$ and $e = \lfloor f \rfloor$, respectively. This operation can be done for several positive 2-cells of $\mathcal{K}$ at once. This simply means that we cut some 2-cells of $\mathcal{K}$ into two parts. The resulting directed 2-complex $\mathcal{K}'$ is called a *subdivision* of $\mathcal{K}$.

Lemma 4.2. *If $\mathcal{K}$ is a directed 2-complex, w is a non-empty 1-path in $\mathcal{K}$ and $\mathcal{K}'$ is a subdivision of $\mathcal{K}$, then the diagram groups $\mathcal{D}(\mathcal{K}, w)$ and $\mathcal{D}(\mathcal{K}', w)$ are isomorphic.*

Proof. We use the notation from the paragraph preceding the formulation of the lemma. Let $\mathcal{K}''$ be the directed 2-complex obtained from $\mathcal{K}$ by adding the edge e and the 2-cells f_1, f_1^{-1}. By part 1) of Theorem 4.1, $\mathcal{D}(\mathcal{K}, w) = \mathcal{D}(\mathcal{K}'', w)$. Now represent $\mathcal{K}''$ as the union of $\mathcal{K}_1$ and $\mathcal{K}_2$, where $\mathcal{K}_1$, $\mathcal{K}_2$ have the same vertices and edges, $\mathcal{K}_1$ contains exactly two 2-cells f, f^{-1}, and $\mathcal{K}_2$ contains all other 2-cells of $\mathcal{K}''$. Notice that $\lceil f \rceil$ is homotopic to e in $\mathcal{K}_2$ (because $\mathcal{K}_2$ contains the 2-cell f_1). Hence by part 2) of Theorem 4.1, we can replace $f^{\pm 1}$ in $\mathcal{K}''$ by $f_2^{\pm 1}$, where f_2 has the form $e = \lfloor f \rfloor$ without changing the diagram group with the base w. But the resulting directed 2-complex is precisely $\mathcal{K}'$. Hence $\mathcal{D}(\mathcal{K}, w) = \mathcal{D}(\mathcal{K}', w)$. □

As an immediate corollary of Lemma 4.2 we get the following statement.

Theorem 4.3. *The classes of diagram groups over semigroup presentations and diagram groups of directed 2-complexes coincide.*

Proof. Notice that complexes of the form $\mathcal{K}_{\mathcal{P}}$ corresponding to semigroup presentations considered in [16] are precisely the directed 2-complexes with one vertex in which different 2-cells cannot have the same top and bottom paths. We have already mentioned that we can only consider directed 2-complexes with one vertex (if we identify vertices we preserve existing diagram groups but the set of diagram groups can increase since the set of 1-paths can increase). It is obvious that if we subdivide each 2-cell of $\mathcal{K}$ twice (into three parts instead of two) then we turn $\mathcal{K}$ into a $\mathcal{K}_{\mathcal{P}}$ for some $\mathcal{P}$. It remains to apply Lemma 4.2. □

5 Morphisms of complexes and universal diagram groups

Let $\mathcal{K}$, $\mathcal{K}'$ be directed 2-complexes. A *morphism* ϕ from $\mathcal{K}$ to $\mathcal{K}'$ is a map that takes vertices to vertices, edges to non-empty 1-paths and 2-cells to 2-paths and preserves the functions ι, τ, $\lceil\cdot\rceil$, $\lfloor\cdot\rfloor$, ${}^{-1}$:

(M1) For every edge e, $\phi(\iota(e)) = \iota(\phi(e))$, $\phi(\tau(e)) = \tau(\phi(e))$,

(M2) For every 2-cell f of $\mathcal{K}$, $\phi(\lceil f \rceil) = \lceil \phi(f) \rceil$, $\phi(\lfloor f \rfloor) = \lfloor \phi(f) \rfloor$; here we set $\phi(p) = \phi(e_1)\phi(e_2)\cdots\phi(e$ for every 1-path $p = e_1 e_2 \cdots e_k$, where e_i are edges (the latter product exists because of (M1)).

(M3) For every 2-cell f of $\mathcal{K}$, $\phi(f^{-1}) = \phi(f)^{-1}$.

Every morphism $\phi: \mathcal{K} \to \mathcal{K}'$ induces a homomorphism $\phi_p: \mathcal{D}(\mathcal{K}, p) \to \mathcal{D}(\mathcal{K}', \phi(p))$ of diagram groups for every 1-path p of $\mathcal{K}$. For every (p,p)-diagram Δ over $\mathcal{K}$, $\phi_p(\Delta)$ is a $(\phi(p), \phi(p))$-diagram obtained from Δ by a) replacing each edge that has label e by a path labelled by $\phi(e)$, and b) replacing each cell that has label f by the diagram over $\mathcal{K}'$ corresponding to the 2-path $\phi(f)$.

For a non-empty 1-path p in $\mathcal{K}$, we say that a morphism $\phi: \mathcal{K} \to \mathcal{K}'$ is *p-nonsingular* if the induced homomorphism ϕ_p is injective on $\mathcal{D}(\mathcal{K}, p)$. In that case the subgroup $\phi_p(\mathcal{D}(\mathcal{K}, p))$ of $\mathcal{D}(\mathcal{K}')$ is called *naturally embedded*. If the morphism ϕ is p-nonsingular for every p, we call it *nonsingular*.

Lemma 5.1. *Let $\phi: \mathcal{K} \to \mathcal{K}'$ be a morphism of two directed 2-complexes.*

1) *If $\phi(\delta)$ is reduced for every reduced 2-path δ, and $\phi(f)$ is not empty for every 2-cell f of $\mathcal{K}$, then ϕ is nonsingular.*

2) *Suppose that ϕ is injective on the set of 2-cells and $\phi(f)$ is a 2-cell for every 2-cell f of $\mathcal{K}$. Then ϕ is nonsingular.*

3) *If a directed 2-complex $\mathcal{K}'$ is obtained by adding 2-cells to a directed 2-complex $\mathcal{K}$, then diagram groups of the form $\mathcal{D}(\mathcal{K}, p)$ are naturally embedded into the diagram groups $\mathcal{D}(\mathcal{K}', p)$, for every 1-path p.*

4) *If a directed 2-complex $\mathcal{K}'$ is obtained by adding 2-cells to a directed 2-complex $\mathcal{K}$, and $\lceil f \rceil$ is homotopic to $\lfloor f \rfloor$ in $\mathcal{K}$ for every 2-cell $f \in \mathcal{K}' \setminus \mathcal{K}$, then for every 1-path p in $\mathcal{K}$, the diagram group $\mathcal{D}(\mathcal{K}, p)$ is a retract of the diagram group $\mathcal{D}(\mathcal{K}', p)$.*

Proof. 1) Indeed, suppose that the kernel of ϕ_p is not trivial. Then it contains a reduced 2-path $\delta \neq \varepsilon(p)$ by Theorem 2.5. By the assumption, $\phi(\delta)$ is reduced, and non-empty, so $\phi_p(\delta) \neq 1$ by Theorem 2.5, a contradiction.

2) It is easy to see that for every reduced (p,p)-diagram Δ, the diagram $\phi_p(\Delta)$ does not contain dipoles. It remains to use part 1) of this lemma.

3) Immediately follows from 2).

4) The retraction ψ is given as follows. Fix a $(\lceil f \rceil, \lfloor f \rfloor)$-diagram Γ_f over $\mathcal{K}$ for every 2-cell $f \in \mathcal{K}' \setminus \mathcal{K}$ in such a way that inverse diagrams correspond to inverse 2-cells. Then for every (p,p)-diagram Δ over $\mathcal{K}'$, the diagram $\psi(\Delta)$ is obtained from Δ by inserting Γ_f instead of every cell in Δ labelled by $f \in \mathcal{K} \setminus \mathcal{K}'$. Clearly, $\psi^2 = \psi$. □

Let us call a directed 2-complex $\mathcal{K}$ *universal* if every finite or countable directed 2-complex maps nonsingularly into $\mathcal{K}$. A diagram group is called *universal* if it contains copies of all countable diagram groups. A directed 2-complex is said to be *2-path connected* if all its non-empty 1-paths are homotopic to each other. By Theorem 3.3 it has at most one non-trivial diagram group up to isomorphism. Notice that if a universal directed 2-complex is 2-path connected then its nontrivial diagram group is universal because every at most countable diagram group is (obviously) a diagram group of at most countable directed 2-complex.

Lemma 5.2. *Let $\mathcal{U}$ be the directed 2-complex*

$$\langle\, x \mid x^m = x^n, x^m = x^n, \ldots \text{ for all } 1 \leq m < n \,\rangle$$

(every equality appears countably many times). Then $\mathcal{U}$ is universal.

Proof. Let $\mathcal{K}$ be at most countable directed 2-complex and let $\mathcal{K}_1$ be obtained from $\mathcal{K}$ by identifying all its edges and vertices. By ϕ we denote the natural morphism from $\mathcal{K}$ to $\mathcal{K}_1$. Then ϕ is nonsingular by Lemma 5.1, part 2. It is easy to see that $\mathcal{U}$ can be obtained from $\mathcal{K}_1$ by adding 2-cells. Hence $\mathcal{K}_1$ is a subcomplex of $\mathcal{U}$ and $\phi\colon \mathcal{K} \to \mathcal{U}$ is nonsingular by Lemma 5.1, part 3. □

We can simplify the universal directed 2-complex $\mathcal{U}$ by using part 2) of Theorem 4.1. For every $1 \le n \le \infty$ let

$$\mathcal{H}_n = \langle\, x \mid x^2 = x, x = x, x = x, \dots \quad (n \text{ times})\,\rangle.$$

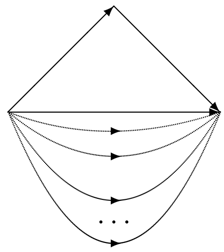

Figure 8.

This complex contains one vertex, one edge, and $n+1$ positive 2-cells, one of which is the Dunce hat and all others are spheres. It is obtained from the plane diagram on Figure 8 by identifying all edges according to their directions.

For example, the complex $\mathcal{H}_0$ is the Dunce hat (see Figure 1).

Lemma 5.3. *The directed 2-complex $\mathcal{H}_\infty$ is universal. In particular, every countable diagram group embeds into the diagram group $\mathcal{D}(\mathcal{H}_\infty, x)$.*

Proof. In fact we shall show that $\mathcal{D}(\mathcal{U}, x)$ is isomorphic to $\mathcal{D}(\mathcal{H}_\infty, x)$. Let us denote by $\mathcal{Q}$ the complex obtained from $\mathcal{U}$ by removing the cell $x^2 = x$ and its inverse. Then $\mathcal{H}_0 \cup \mathcal{Q} = \mathcal{U}$. It is easy to see that every non-empty 1-path in $\mathcal{H}_0$ is homotopic to x. Therefore, let us replace each 2-cell $x^m = x^n$ in $\mathcal{U}$ by a cell $x = x$, and obtain a complex $\mathcal{H}_\infty$. By part 2) of Theorem 4.1, the diagram groups of $\mathcal{U}$ and $\mathcal{H}_\infty$ are isomorphic. The proof of part 2) of Theorem 4.1 actually gives us a nonsingular morphism from $\mathcal{U}$ into $\mathcal{H}_\infty$. It remains to use Lemma 5.2. □

The diagram group $\mathcal{D}(\mathcal{H}_\infty, x)$ is not even finitely generated (see Example 6.8 below).

Our next goal is to show that already the group $\mathcal{D}(\mathcal{H}_1, x)$ is universal. This will follow from Lemma 5.3 and the fact that there exists a nonsingular morphism from $\mathcal{H}_\infty$ to $\mathcal{H}_1$. The group $\mathcal{D}(\mathcal{H}_1, x)$ is finitely presented (see Example 6.8) and has a nice structure (see [19]).

Lemma 5.4. *There exists a nonsingular morphism from $\mathcal{H}_\infty$ to $\mathcal{H}_2$.*

Proof. Let us label the positive 2-cells of $\mathcal{H}_\infty$ by f_0, f_1, ..., where f_0 is the cell $x^2 = x$. The complex $\mathcal{H}_2$ has positive 2-cells f_0, f_1, f_2. Consider the morphism ϕ from $\mathcal{H}_\infty$ to $\mathcal{H}_2$ which takes the edge x to x, f_0 to f_0, and each f_i, $i \ge 1$, to the 2-path

$$(1, f_1, 1)^i \circ (1, f_2, 1) \circ (1, f_1, 1)^i. \tag{7}$$

We need to show that ϕ_x is injective (then every ϕ_{x^k} will be injective too because all local groups in the diagram groupoid $\mathcal{D}(\mathcal{H}_\infty)$ are conjugate by Corollary 3.4).

Indeed, let Δ be a nontrivial reduced diagram over $\mathcal{H}_\infty$. Then $\bar{\Delta} = \phi_x(\Delta)$ is obtained by replacing every cell labelled by f_i by the diagram Γ_i corresponding to the 2-path (7) (see Figure 9) and each cell labelled by f_i^{-1} by the diagram Γ_i^{-1}.

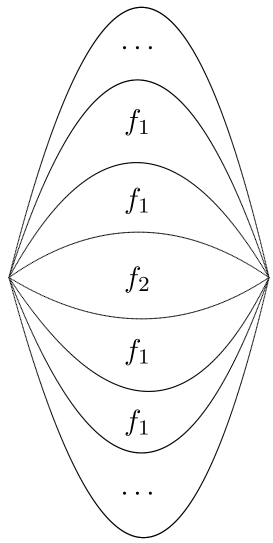

Figure 9.

It is sufficient to show that $\bar{\Delta}$ is also nontrivial. Any diagram over $\mathcal{H}_n$ (for any n) can be uniquely decomposed into subdiagrams of the following two types. Each subdiagram of the first type is an (x^2, x)-cell or its mirror image. Each subdiagram of the second type is a maximal (x, x)-subdiagram, which is a product of (x, x)-cells only.

If we decompose $\bar{\Delta}$ in such a way, then we see that each subdiagram of the second type is the image of a subdiagram of the second type in Δ under the mapping ϕ_x. Since Δ has no dipoles, each of its subdiagram of the second type is a product of (x, x)-cells, where the word formed by their labels is a freely irreducible group word v over the alphabet $\{f_i^{\pm 1} \mid i \geq 1\}$. Notice that the ϕ_x-image of this subdiagram is a product of (x, x)-cells such that the word formed by their inner labels is $\bar{v}$, where $\bar{f}_i = f_1^i f_2 f_1^i$.

Since the map $f_i \mapsto \bar{f}_i$ is an embedding of the free group with generators f_i $(i \geq 1)$ into the free group with two generators f_1, f_2, the word $\bar{v}$ will be non-empty after all free cancellations are made in it. Thus if we reduce dipoles in each subdiagram of the second type in $\bar{\Delta}$, the resulting diagram Δ' will be reduced. Indeed, we have cancelled all the dipoles formed by (x, x)-cells. No dipoles between cells that correspond to $x^2 = x$ may appear because each subdiagram of the second type in $\bar{\Delta}$ remains nontrivial after all cancellations. The diagram Δ' contains the same number of cells labelled by $f_0^{\pm 1}$ as Δ and at least as many (x, x)-cells as Δ. Hence Δ' is nontrivial. □

Lemma 5.5. *There exists a nonsingular morphism from $\mathcal{H}_2$ into $\mathcal{H}_1$.*

Proof. The idea is similar to the one of the previous lemma. We keep notation for 2-cells of $\mathcal{H}_2$ from that lemma. Letters f_i $(i = 1, 2)$ will be also used to denote the atomic 2-paths $(1, f_i, 1)$ and the corresponding (x, x)-diagrams that consist of one cell labelled by f_i. By a we denote any nontrivial reduced (x, x)-diagram over $\mathcal{H}_0 = \langle x \mid x^2 = x \rangle$ and one of the corresponding 2-paths in $\mathcal{H}_0$. Any two (x, x)-diagrams can be concatenated. So each word in $f_1^{\pm 1}$, $f_2^{\pm 1}$, $a^{\pm 1}$ denotes some (x, x)-diagram.

Now we use the morphism $\psi: \mathcal{H}_2 \to \mathcal{H}_1$ that takes the edge x to x, f_1 to $f_1 a f_1$, f_2 to $f_1^2 a f_1^2$.

Let Δ be a nontrivial reduced diagram over $\mathcal{H}_2$. Let $\tilde{\Delta}$ be the diagram obtained from $\psi_x(\Delta)$ by cancelling all dipoles of (x, x)-cells. As in Lemma 5.4, we subdivide Δ into subdiagrams of the two types. Let Γ be a subdiagram of the second type. It is a product of cells with inner labels $f_1^{\pm1}$, $f_2^{\pm1}$. Let v be the word that is the product of these labels. Clearly, this is a freely reduced word in $f_1^{\pm1}$, $f_2^{\pm1}$. After we replace v by $\bar{v}$, where $\bar{f}_1 = f_1 a f_1$, $\bar{f}_2 = f_1^2 a f_1^2$ and then freely reduce the result, we get a word of the form

$$f_1^{s_0} a^{k_1} f_1^{s_1} \cdots a^{k_r} f_1^{s_r}, \tag{8}$$

where r is the length of v. Note that $s_0 \neq 0$, $s_r \neq 0$, $k_1, \ldots, k_r = \pm 1$. Note also that none of the occurrences of letters $a^{\pm1}$ in $\bar{v}$ disappears after the reduction, hence only occurrences of the letter $f_1^{\pm1}$ can disappear. Thus the ψ_x-image of any subdiagram of the second type after cancelling all dipoles of (x, x)-cells becomes reduced and of the form (8) as well. After we cancel all (x, x)-dipoles in the subdiagrams $\psi_x(\Gamma)$ for all maximal subdiagrams Γ of Δ of the second type, we would not have any more (x, x)-dipoles. Hence we shall get the diagram $\tilde{\Delta}$.

Since s_0 and s_r are always non-zero, the 2-cells labelled by $f_0^{\pm1}$ cannot form a dipole in $\tilde{\Delta}$. Therefore, $\tilde{\Delta}$ is reduced. Since the number of cells in $\tilde{\Delta}$ is at least the same as in Δ, the diagram $\tilde{\Delta}$ is nontrivial. □

Theorem 5.6. *The directed* 2*-complex* $\mathcal{H}_1$ *is universal. Hence the group* $\mathcal{G}_1 = \mathcal{D}(\mathcal{H}_1, x)$ *contains copies of all countable diagram groups. This group is finitely presented and even of type* $\mathcal{F}_\infty$*.*

Proof. The first statement follows from Lemmas 5.3, 5.4, 5.5. The fact that the group $\mathcal{G}_1$ is of type $\mathcal{F}_\infty$ follows from Theorem 7.3 below. It can also be deduced from results of [14]. In Example 6.8 below, we shall compute a presentation of $\mathcal{G}_1$. It has 6 generators and 18 defining relations. □

Theorem 5.6 gives an example of a universal directed 2-complex with one edge and two positive 2-cells. The following theorem shows that a complex with one edge and one positive 2-cell can be universal as well.

Theorem 5.7. *The directed* 2*-complex* $\mathcal{V} = \langle\, y \mid y^3 = y^2 \,\rangle$ *is universal* (*this complex is obtained from Figure* 10 *by identifying all edges according to their directions*)*. Its diagram group* $\mathcal{D}(\mathcal{V}, y^2)$ *is also a universal group of type* $\mathcal{F}_\infty$*. It has the following Thompson-like presentation:*

$$\langle\, x_0, x_1, \ldots, y_0, y_1, \ldots \mid x_j^{x_i} = x_{j+1}, y_j^{x_i} = y_{i+1},\ 0 \le i < j-1 \,\rangle$$

Proof. Let us consider the two diagrams A and B over $\mathcal{V}$ on Figure 10. Here A is a (y^8, y^4)-diagram and B is a (y^4, y^4)-diagram.

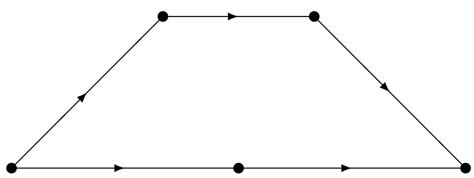

Figure 10.

These diagrams correspond to the following 2-paths on $\mathcal{V}$. Let $p_{i,j}$ $(i, j \ge 0)$ denote the atomic 2-path $(y^i, y^3 = y^2, y^j)$. Then A corresponds to the 2-path $\alpha = p_{33}^{-1} p_{15} p_{41} p_{04} p_{30} p_{11}$, whereas B corresponds to $\beta = p_{01} p_{10}^{-1}$. Let us consider the morphism γ from $\mathcal{H}_1$ to $\mathcal{V}$ which takes the edge x to the 1-path y^4, the 2-cell f_0 to α, and the 2-cell f_1 to β. We are going to show that γ is nonsingular. As before, it is enough to show that it is x-nonsingular.

Let Δ be any reduced (x,x)-diagram over $\mathcal{H}_1$. The diagram $\gamma_x(\Delta)$ is obtained as follows. First we subdivide each edge labelled by x into 4 parts and label each of them by y. Then each (x^2,x)-cell becomes a (y^8,y^4)-cell. Every (x^2,x)-cell with inner label f_0 is replaced by A. A mirror image of such a cell is replaced by A^{-1}. Similarly, any (x,x)-cell of Δ becomes a (y^4,y^4)-cell, so we replace by $B^{\pm 1}$ all (x,x)-cells labelled by $f_1^{\pm 1}$. After all these replacements, we get a (y^4,y^4)-diagram $\hat{\Delta}$ over $\mathcal{V}$.

We shall show that $\hat{\Delta}$ is reduced, and then apply Lemma 5.1, part 1). Note that each of the diagrams A, B has no dipoles so a dipole in $\hat{\Delta}$, if it occurs, must belong to different subdiagrams of the form $A^{\pm 1}$, $B^{\pm 1}$. Suppose that the upper cell of the dipole is contained in $A^{\pm 1}$. From the structure of A it is obvious that this cell must be a (y^3,y^2)-cell. So it cannot form a dipole with a cell from $B^{\pm 1}$. The lower cell of the dipole is a (y^2,y^3)-cell.

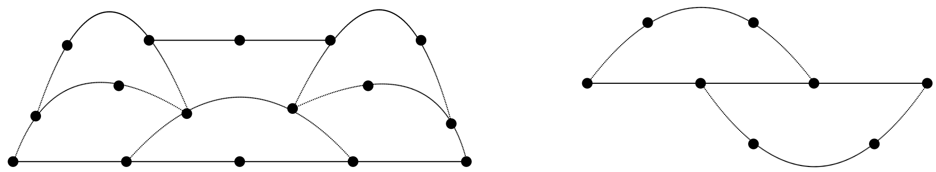

Figure 11.

There are two types of vertices in $\hat{\Delta}$. Vertices of the first type (we call them *red*) are the images of vertices of Δ. The other vertices are called *green*. It is easy to see that B has only two red vertices, $\iota(B)$ and $\tau(B)$. The diagram A has exactly three red vertices: $\iota(A)$, $\tau(A)$, and the middle point of the top path of A. The middle point of the bottom path of A is green. So we have to consider two cases for the two cells that form the dipole. In the first case the middle point of the common part of the boundary of the cells forming a dipole is red, in the second case it is green.

In the first case the upper cell of the dipole is contained in a copy of A^{-1} and the lower cell is contained in a copy of A. Denote these copies by Γ_1, Γ_2, respectively. We claim that the bottom path of Γ_1 coincides with the top path of Γ_2. Indeed, Γ_1 is the γ-image of an (x,x^2)-cell π_1 in Δ and Γ_2 is the γ image of an (x^2,x)-cell π_2 in Δ. The bottom path of π_1 was subdivided into 8 parts. The same is true for the top path of π_2. The product of the 4th and the 5th of these parts is the same for both π_1 and π_2 since it is the common boundary of the cells forming the dipole. This can happen only if the bottom path of π_1 coincides with the top path of π_2. But in this case we have a dipole in Δ. This contradicts the assumption that Δ is reduced.

In the second case the upper cell of the dipole is contained in a copy of A and the lower cell is contained in a copy of A^{-1}. We also denote these copies by Γ_1, Γ_2, respectively. The bottom path of Γ_1 is the image of an edge in Δ. This edge is subdivided into 4 parts. The product of its second and third part is the common boundary of the cells of the dipole. The same is true for the top path of Γ_2. So the bottom of Γ_1 coincides with the top of Γ_2 since they must be images of the same edge in Δ. In this case an (x^2,x)-cell forms a dipole in Δ with an (x,x^2)-cell, a contradiction.

If the lower cell of the dipole is contained in $A^{\pm 1}$, then the same arguments are applied. So to finish the proof, let us assume that the dipole in $\hat{\Delta}$ is formed by two cells that are contained in copies of $B^{\pm 1}$. Suppose that the upper cell of the dipole belongs to a copy of B. Thus it is a (y^2,y^3)-cell. Note that the leftmost point of it is green and the rightmost point is red. The lower cell must be a (y^3,y^2)-cell with the corresponding points of the same colour. Thus the lower cell is contained in a copy of B^{-1}. As in the previous paragraph, we see from this fact that the last 3 of 4 sections of some edges in Δ coincide. Then these edges also coincide and so Δ has a dipole that consists of two (x, x)-cells. The case when the upper cell of the dipole belongs to a copy of B^{-1} is quite analogous. Thus γ is nonsingular.

That diagram groups of $\mathcal{V}$ are of type $\mathcal{F}_\infty$ follows directly from [14]. The presentation of $\mathcal{D}(\mathcal{V}, y^2)$ is found in [16, page 114]. It can be found using Theorem 6.5 below. $\square$

In the next section we shall construct a universal 2-complex whose diagram groups have very simple presentations.

6 Presentations of diagram groups

In [16, Section 9], we showed how to find nice presentations of diagram groups of the so called complete string rewriting systems. Here we shall generalize these results for diagram groups of directed 2-complexes.

We start with a definition of a complete directed 2-complex. Throughout this section, $\mathcal{K}$ is a directed 2-complex with the set of edges $\mathbf{E}$, set of 2-faces $\mathbf{F}$ and a fixed set of positive faces $\mathbf{F}^+$.

Let p, q be 1-paths in $\mathcal{K}$. We write $p \xrightarrow{+} q$ if $p \neq q$ and there exists a positive 2-path δ with $\lceil\delta\rceil = p$ and $\lfloor\delta\rfloor = q$.

We say that $\mathcal{K}$ is *Noetherian* if every sequence of 1-paths $p_1 \xrightarrow{+} p_2 \xrightarrow{+} \cdots$ terminates.

We say that $\mathcal{K}$ is *confluent*, if for every two positive 2-paths δ_1, δ_2 with $\lceil\delta_1\rceil = \lceil\delta_2\rceil$ there exist two positive 2-paths $\delta_1 \circ \delta_1'$ and $\delta_2 \circ \delta_2'$ such that $\lfloor\delta_1'\rfloor = \lfloor\delta_2'\rfloor$. In that case we say that δ_1 and δ_2 can be *extended to a diamond*.

If a directed 2-complex is Noetherian and confluent, then we say that $\mathcal{K}$ is *complete*.

It is easy to see that if $\mathcal{K} = \mathcal{K}_\mathcal{P}$ for some complete string rewriting system $\mathcal{P}$, then $\mathcal{K}$ is complete.

Let δ_1, δ_2 be two positive atomic 2-paths on $\mathcal{K}$. Assume that $\lceil\delta_i\rceil \neq \lfloor\delta_i\rfloor$ for each $i = 1, 2$. Suppose that one of the two cases hold:

1. $\delta_1 = (1, f_1, q)$, $\delta_2 = (p, f_2, 1)$, where $\lceil f_1\rceil = ps$, $\lceil f_2\rceil = sq$ for some non-empty 1-path s;
2. $\lceil f_2\rceil$ is a subpath of $\lceil f_1\rceil$ and $f_1 \neq f_2$.

Then we say that δ_1 and δ_2 form a *critical pair*. The diagrams representing these cases are shown on Figure 12.

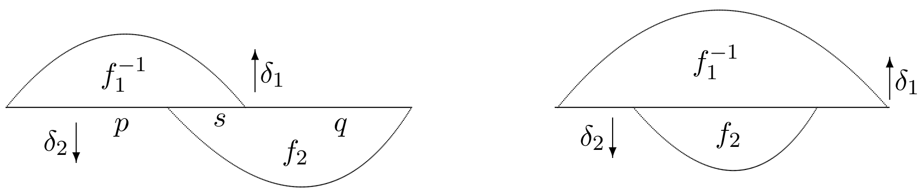

Figure 12.

We say that the critical pair can be *resolved* if it can be extended to a diamond.

For string rewriting systems, it is known (Newman's lemma [16]) that a Noetherian string rewriting system is complete if and only if every critical pair can be resolved. One can similarly prove that a Noetherian directed 2-complex is complete if and only if every critical pair of its positive atomic 2-paths can be resolved.

A 1-path p in a complete directed 2-complex $\mathcal{K}$ is called *irreducible* if $p \xrightarrow{*} q$ is impossible (i.e. p cannot be changed by any positive 2-path). It is easy to see that every 1-path p in a complete directed 2-complex $\mathcal{K}$ is homotopic to a unique irreducible 1-path $\bar{p}$, which is called the *irreducible form* of p.

From Lemma 5.1, part 4), one can almost immediately deduce the following important statement.

Lemma 6.1. *Every diagram group of a directed* 2*-complex* $\mathcal{K}$ *is a retract of a diagram group of a complete directed* 2*-complex* $\mathcal{K}' \supseteq \mathcal{K}$*. The number of classes of homotopic* 1*-paths in* $\mathcal{K}'$ *is the same as in* $\mathcal{K}$*. If* $\mathcal{K}$ *is finite and it has finitely many classes of homotopic* 1*-paths, then* $\mathcal{K}'$ *is also finite.*

Proof. Let $G = \mathcal{D}(\mathcal{K}, p)$. Let us fix some total well ordering on the set of edges of $\mathcal{K}$. Then we can introduce the ShortLex order on 1-paths of $\mathcal{K}$.

Let us change the orientation on the set of 2-cells of $\mathcal{K}$ as follows. For every positive cell $f \in \mathcal{K}$, if $\lceil f \rceil$ is smaller than $\lfloor f \rfloor$ in ShortLex, we call f negative and f^{-1} positive. This operation does not change the diagram groups of the 2-complex and the classes of homotopic 1-paths (see Remark 2.2).

In every class W of homotopic 1-paths of $\mathcal{K}$ choose the ShortLex smallest 1-path $p(W)$. Now add cells to $\mathcal{K}$ as follows. First for every edge e in $\mathcal{K}$, we add a positive 2-cell f_e of the form $e = p(W)$, where W is the class of homotopic 1-paths containing e. We also add the inverse of that 2-cell. Now let U and V be two classes of homotopic 1-paths in $\mathcal{K}$ such that the product $p(U)p(V)$ exists (that is, $\tau(p(U)) = \iota(p(V))$) and let W be the class of homotopic 1-paths that contains $p(U)p(V)$. Add a positive cell $f_{U,V}$ with top path $p(U)p(V)$ and bottom path $p(W)$ (also add the corresponding negative 2-cell). As a result of these operations, the number of classes of homotopic 1-paths does not change.

The resulting complex $\mathcal{K}'$ is clearly Noetherian. Indeed, for every positive 2-cell f of $\mathcal{K}'$, $\lceil f \rceil \geq \lfloor f \rfloor$ in the ShortLex order.

Every 1-path of a class W is connected to $p(W)$ by a positive 2-path consisted of atomic 2-paths corresponding to the cells of the form f_e and $f_{U,V}$. (This can be easily proved by induction on the length of the 1-path.) Hence the complex $\mathcal{K}'$ is confluent. Indeed, for every two positive atomic 2-paths δ_1, δ_2 in $\mathcal{K}$ with $\lceil \delta_1 \rceil = \lceil \delta_2 \rceil$, their bottom 1-paths are homotopic. So one can reduce each of them to the same 1-path and complete the diamond. Thus $\mathcal{K}'$ i s a complete directed 2-complex.

The complex $\mathcal{K}'$ is also confluent. Indeed, for every two positive atomic 2-paths δ_1, δ_2 in $\mathcal{K}$ with $\lceil \delta_1 \rceil = \lceil \delta_2 \rceil$, the 1-paths $\lfloor \delta_1 \rfloor$ and $\lfloor \delta_2 \rfloor$ are in the same class W of homotopic 1-paths. Now using the new cells f_e and $f_{U,V}$, one can reduce each of these 1-paths to $p(W)$ and complete the diamond. Thus $\mathcal{K}'$ is a complete directed 2-complex.

By Lemma 5.1, part 4), G is a retract of $\mathcal{D}(\mathcal{K}', p)$. The last statement of the theorem obviously holds because we add only finitely many cells. ☐

Since we are looking for nice presentations of diagram groups, which are fundamental groups of Squier complexes, it is natural to start with finding nice spanning forests in $\mathrm{Sq}(\mathcal{K})$. It can be done in the case when $\mathcal{K}$ is complete (and in some other cases which we do not discuss here). Recall that a *spanning forest* of a 1-complex $\mathcal{S}$ is a forest whose intersection with every connected component of $\mathcal{S}$ is a spanning tree in that component.

Definition 6.2. Let $\mathcal{K}$ be a complete directed 2-complex. A spanning forest T in $\mathrm{Sq}(\mathcal{K})$ is called a *left forest* whenever the following two conditions hold:

(F1) for any edge $e = (p, f, q)$ in T, the 1-path p is irreducible;

(F2) if an edge $e = (p, f, q)$ belongs to T, then any edge of the form (p, f, q') also belongs to T.

Analogously one can define a *right forest*.

Because of the property (F2), we will often use the notation $(u, f, *)$ when we mention an edge of a left forest. Analogously, $(*, f, v)$ will be used for edges from a right forest.

Lemma 6.3. *If $\mathcal{K}$ is a complete directed 2-complex, then $\mathrm{Sq}(\mathcal{K})$ has a left forest and a right forest.*

Proof. In each connected component of $\mathrm{Sq}(\mathcal{K})$ we choose the vertex which is an irreducible 1-path. If p is not irreducible, then we find its shortest initial segment p', which is not irreducible. Let $p = p'v$ for some v. By definition, p' can be reduced so it has a subpath of the form $\lfloor f \rfloor$ for some negative cell f of $\mathcal{K}$, where $\lfloor f \rfloor \neq \lceil f \rceil$ (there are possibly many ways to choose f with the above properties but we choose **one** of them arbitrarily). Obviously, this subpath is a suffix of p'. Hence $p' = u\lfloor f \rfloor$, where u must be irreducible. Thus to every vertex p of $\mathrm{Sq}(\mathcal{K})$ we can assign an edge $e = (u, f, v)$. Let us consider the subgraph T of the 1-skeleton of $\mathrm{Sq}(\mathcal{K})$ that contains all vertices and all the edges of the form $e^{\pm 1}$, where e was assigned to some p. We leave it as an exercise for the reader to check that T is a spanning forest and that it satisfies conditions (F1), (F2) (see the proof of [16, Lemma 9.4]). A right forest in $\mathrm{Sq}(\mathcal{K})$ is constructed in a similar way. □

Remark 6.4. Let $\mathcal{K}$ be a complete directed 2-complex. In general, the way to construct a left (right) forest from the proof of Lemma 6.3 is not unique. However, if the second case of the critical pair from its definition never occurs (that will be the case in all the examples considered below), then it is not difficult to prove that the left forest in $\mathrm{Sq}(\mathcal{K})$ is unique and consists of all edges $(u, f, v)^{\pm 1}$, where $f \in \mathbf{F}^-$, $\lfloor f \rfloor \neq \lceil f \rceil$, and every proper initial subpath of $u\lfloor f \rfloor$ is irreducible.

Let us fix a left forest T_l and a right forest T_r in $\mathrm{Sq}(\mathcal{K})$. Then for every vertex p in $\mathrm{Sq}(\mathcal{K})$, where $p \neq \bar{p}$, there exists a unique negative edge $e \in T_l$ (resp., $e \in T_r$) going into p. Indeed, otherwise there would be two different paths in T_l (resp., T_r) that consist of positive edges and connect p with $\bar{p}$. We shall say that e is *assigned* to p.

The following theorem is a translation of [16, Theorem 9.5]. It gives a Wirtinger-like presentation of any diagram group of a complete directed 2-complex. The translation of the proof from [16] in straightforward.

Theorem 6.5. *Let $\mathcal{K}$ be a complete directed 2-complex, with the set of negative 2-cells $\mathbf{F}^-$, and a distinguished non-empty 1-path w. Then the diagram group $\mathcal{D}(\mathcal{K}, w)$ admits the following presentation. The generating set S consists of all the negative edges in $\mathrm{Sq}(\mathcal{K}, w)$ excluding edges from the left forest T_l. The defining relations are all relations of the form*[3]

$$(p, f_1, q\lfloor f_2 \rfloor r) = (p, f_1, q\lceil f_2 \rceil r)^{(\overline{p\lceil f_1 \rceil q}, f_2, r)} \tag{9}$$

if the edge $e = (\overline{p\lceil f_1 \rceil q}, f_2, r)$ is not in T_l, or of the form

$$(p, f_1, q\lfloor f_2 \rfloor r) = (p, f_1, q\lceil f_2 \rceil r) \tag{10}$$

if $e \in T_l$. Here $f_1, f_2 \in \mathbf{F}^-$, and all edges involved in these relations are from the generating set S.

[3] Here and below x^y means $y^{-1}xy$.

In most cases, this presentation can be simplified.

As in [16], with every negative edge (u, f, v) in $\mathrm{Sq}(\mathcal{K})$ we associate a group word $[u, f, v]$ in the alphabet of negative edges of $\mathrm{Sq}(\mathcal{K})$ and their inverses, defined by the Noetherian induction on the strict order generated by $\xrightarrow{+}$ and the relation suff, where $(u, u') \in$ suff if and only if u' is a proper suffix of u:

- If $u \neq \bar{u}$, then $[u, f, v] = [\bar{u}, f, v]$.
- If $u = \bar{u}$ and (u, f, v) is in T_l, then $[u, f, v] = 1$.
- If $u = \bar{u}$, $v = \bar{v}$ and (u, f, v) is not in T_l, then $[u, f, v] = (u, f, v)$.
- If $u = \bar{u}$, $v \neq \bar{v}$ and (u, f, v) is not in T_l, then take the negative edge (p, g, q) from the right forest T_r that is assigned to v (thus, $g \in \mathbf{F}^-$, $v = p\lfloor g \rfloor q$, and $q = \bar{q}$). By the induction hypothesis, we can assume that the word $[u, f, p\lceil g \rceil q]$ is already defined. Then let

$$[u, f, v] = [u, f, p\lceil g \rceil q]^{[\overline{u\lceil f \rceil p}, g, q]}.$$

Notice that every letter (or its inverse) in any word $[u, f, v]$ has the form (p, g, q), where p, q are irreducible, $g \in \mathbf{F}^-$, and (p, g, q) is not in T_l.

Finally, let us present the translation of [16, Theorem 9.8] into the language of directed 2-complexes (we are correcting some misprints in the formulation of that theorem as well). The translation of the proof of that theorem is straightforward.

Theorem 6.6. *Let $\mathcal{K}$ be a complete directed 2-complex and let w be a non-empty 1-path in $\mathcal{K}$. The group $\mathcal{D}(\mathcal{K}, w)$ is generated by the set X of all edges (u, f, v) in $\mathrm{Sq}(\mathcal{K}, w)$, where u, v are irreducible, $f \in F^-$, and (u, f, v) is not in the left forest T_l, subject to the following defining relations:*

$$[p, f_1, q\lfloor f_2 \rfloor r] = [p, f_1, q\lceil f_2 \rceil r]^{[\overline{p\lceil f_1 \rceil q}, f_2, r]}, \tag{11}$$

where

- *$f_1, f_2 \in F^-$,*
- *p, q, r are irreducible,*
- *$p\lceil f_1 \rceil q\lceil f_2 \rceil r$ is homotopic to w in $\mathcal{K}$,*
- *$(p, f_1, *)$ is not in T_l and $(*, f_2, r)$ is not in T_r.*

Remark 6.7. a) Notice that every relation in Theorem 6.6 is a conjugacy relation of the form $x^y = x^z$ for some generator x and words y, z. Therefore, the set X in Theorem 6.6 is a minimal generating set of the diagram group (because it freely generates the abelianization of $\mathcal{D}(\mathcal{K}, w)$). In Section 7 we will show that the number of defining relations given by Theorem 6.6 is also minimal possible.

b) One can check that in the formulation of Theorem 6.6, we can replace T_r by T_l. The numbers of generators and relations will be the same (see Remark 7.8 below), but the relations in general will be more complicated.

Example 6.8. As we mentioned before, the directed 2-complex $\mathcal{H}_n$ is complete for every n. This complex has only two irreducible 1-paths, 1 and x. Let g_0 be the negative 2-cell of $\mathcal{H}_n$ of the form $x = x^2$ and let $g_1, g_2, \ldots, g_n$ be the negative 2-cells of $\mathcal{H}_n$ of the form $x = x$. Then the left forest consists of edges of the form $(1, g_0, *)^{\pm 1}$. The right forest consists of edges of the form $(*, g_0, 1)^{\pm 1}$.

By Theorem 6.6, the diagram group $\mathcal{D}(\mathcal{H}_n, x)$ is generated by the edges of the forms $(x, g_0, 1)$, (x, g_0, x), and (p, g_i, q), $1 \le i \le n$, where $p, q \in \{1, x\}$ (the number of generators is $4n+2$) subject to the conjugacy relations (11), where there are $2n+1$ choices for the pair (p, f_1), $2n+1$ choices for the pair (f_2, r), and q can be equal to 1 or x (the number of relations is $2(2n+1)^2$).

In particular, if $n = \infty$, the diagram group is not finitely generated. Otherwise the group $\mathcal{D}(\mathcal{H}_n, x)$ is finitely presented. In case $n = 0$, it has two generators $x_0 = (x, g_0, 1)$, $x_1 = (x, g_0, x)$ and two defining relations $x_1^{x_0^2} = x_1^{x_0 x_1}$, $x_1^{x_0^3} = x_1^{x_0^2 x_1}$. This is one of the classical presentations of R. Thompson's group F [13]. In case $n = 1$ we get a presentation of the group $\mathcal{G}_1$ with 6 generators and 18 defining relations.

Now we are going to use Theorems 6.5 and 6.6 to give an example of a universal diagram group with a very simple presentation.

Theorem 6.9. *Let $\mathcal{K} = \langle\, a, y \mid ay = a, y^3 = y^2 \,\rangle$. Then $\mathcal{K}$ is a universal directed 2-complex. The group $H = \mathcal{D}(\mathcal{K}, a)$ is universal. It can be given by the following Thompson-like group presentation*

$$\langle\, x_0, x_1, x_2, \ldots \mid x_j^{x_i} = x_{j+1},\ 0 \le i < j-1 \,\rangle. \tag{12}$$

The group H also has the following finite presentation with three generators and six defining relations:

$$\langle\, x_0, x_1, x_2 \mid x_2^{x_0^i} = x_2^{x_0^{i-1} x_1} (i = 2, 3, 4), x_2^{x_0^j} = x_2^{x_0^{j-1} x_2} (j = 3, 4, 5) \,\rangle. \tag{13}$$

Proof. It is easy to see that $\mathcal{K}$ contains $\mathcal{V}$, whence $\mathcal{K}$ is universal, and $\mathcal{D}(\mathcal{V}, y^2)$ is embedded into $\mathcal{D}(\mathcal{K}, y^2)$. Hence $\mathcal{D}(\mathcal{K}, y^2)$ contains copies of all countable diagram groups. By Corollary 3.5, $\mathcal{D}(\mathcal{K}, a) \times \mathcal{D}(\mathcal{K}, y^2)$ is embedded into $\mathcal{D}(\mathcal{K}, ay^2)$. Since ay^2 and a are homotopic in $\mathcal{K}$, by Corollary 3.4, we can conclude that $\mathcal{D}(\mathcal{K}, y^2)$ is embedded into $\mathcal{D}(\mathcal{K}, a)$. Hence $\mathcal{D}(\mathcal{K}, a)$ also contains copies of all countable diagram groups.

It is easy to check that the directed 2-complex $\mathcal{K}$ is complete.

Theorem 6.5 implies that H can be generated by the edges of the form $(u, a = ay, v)$ or $(u, y^2 = y^3, v)$, where u, v are 1-paths in $\mathcal{K}$, u is irreducible, and uav (resp., uy^2v) is homotopic to a in $\mathcal{K}$.

If uav is homotopic to a, then clearly u is empty, which implies that $(u, a = ay, v)$ is in the left forest. Hence H is generated by the edges of the form $(u, y^2 = y^3, v)$ only. If $(u, y^2 = y^3, v)$ is one of our generators, then uy^2v must be homotopic to a. Then $u = ay^k$, $v = y^l$ for some $k, l \ge 0$. Since u must be an irreducible 1-path, we have $k = 0$. So this generator has the form $(a, y^2 = y^3, y^l)$. We denote it by x_l. By Theorem 6.5 the defining relations of H are the following

$$(a, y^2 = y^3, vy^3w) = (a, y^2 = y^3, vy^2w)^{(\overline{ay^2v}, y^2 = y^3, w)}.$$

Note that $\overline{ay^2v} = a$. Let $i = |w|$, $j = |vy^2w| = i + 2 + |v|$. Then our defining relation has the form $x_{j+1} = x_j^{x_i}$, where $j \ge i+2$. This leads to (12).

To describe a finite presentation of H, we use Theorem 6.6. Our set of generators now consists of the elements $x_j = (a, y^2 = y^3, y^j)$, where $j = 0, 1, 2$ since the word y^j is irreducible.

The defining relations have the form

$$[a, y^2 = y^3, py^3q] = [a, y^2 = y^3, py^2q]^{[a,y^2=y^3,q]}$$

since $\overline{ay^2p} = a$, where the words p, q are irreducible. Also we have a restriction that $(a, y^2 = y^3, q)$ is not in the right forest. Hence q is non-empty. Therefore $p = 1, y, y^2$ and $q = y, y^2$. Let $z_j = [a, y^2 = y^3, y^j]$. According to Definition 6.2, z_j can be expressed as follows in terms of the generators: $z_0 = (a, y^2 = y^3, 1) = x_0$, $z_1 = x_1$, $z_2 = x_2$, $z_3 = [a, y^2 = y^3, y^3] = [a, y^2 = y^3, y^2]^{[a,y^2=y^3,1]} = x_2^{x_0}$ and analogously $z_4 = z_3^{z_0} = x_2^{x_0^2}$, $z_5 = x_2^{x_0^3}$, and so on. Thus we have 6 defining relations in terms of the z_j's: $z_4 = z_3^{z_1}$, $z_5 = z_4^{z_1}$, $z_6 = z_5^{z_1}$, $z_5 = z_4^{z_2}$, $z_6 = z_5^{z_2}$, $z_7 = z_6^{z_2}$. If we now rewrite them in terms of the generators x_0, x_1, x_2, we obtain (13). □

7 Homology

Theorem 3.6 shows that the components $\mathrm{Sq}(\mathcal{K}, w)$ are $K(G, 1)$ spaces for diagram groups $G = \mathcal{D}(\mathcal{K}, w)$. In fact in most cases $\mathrm{Sq}(\mathcal{K})$ is too large. Here we will use the technique of collapsing schemes [9, 8, 12] to find a "smaller" CW complex, which is homotopy equivalent to $\mathrm{Sq}(\mathcal{K})$ (at least in the case when $\mathcal{K}$ is complete).

We recall the concept of collapsing scheme from [8, 12, 9]. Let X be a semi-cubical complex. We say that we have a collapsing scheme for X if the following is true:

- there exists a subdivision of the set of all cubes of X into three disjoint subsets: *essential*, *collapsible*, and *redundant* cubes;
- there exists a strict partial order $\succ$ on the set of all redundant n-cubes ($n \geq 0$) that satisfies the descending chain condition (that is, any sequence $c_1 \succ c_2 \succ \cdots$ terminates).
- there exists a bijection $c \mapsto \hat{c}$ between the set of all redundant n-cubes and the set of all collapsible $(n+1)$-cubes (for every $n \geq 0$);
- any redundant n-cube c occurs exactly once among the n-faces of $\hat{c}$ and all the other redundant n-faces c' of $\hat{c}$ precede c in the order $\succ$ (that is, $c \succ c'$); the redundant n-cube c is called the *free face* of the collapsible $(n+1)$-cube $\hat{c}$.

The next lemma is proved in almost the same way as [8, Proposition 1].

Lemma 7.1. *Given a collapsible scheme for a semi-cubical complex X, one can construct a CW complex Y which is homotopy equivalent to X in such a way that the n-cells of Y are in one-to-one correspondence with the essential n-cubes of X.*

As in [8], for each $n \geq 0$ one has to do an infinite number of elementary steps, one for each collapsible n-cube. The free face of a collapsible cube is identified (homeomorphically) with the union of the other faces and the collapsible cube disappears. The space Y is a result of the whole process.

Now let $\mathcal{K}$ be a complete directed 2-complex. Let T_l be a left forest in $\mathcal{K}$. Recall that for any 1-path p, the irreducible form of p is denoted by $\bar{p}$. Now let us construct a collapsing scheme for the semi-cubical complex $X = \mathrm{Sq}(\mathcal{K})$. Let $c = \varepsilon(u_0) + f_1 + \cdots + f_n + \varepsilon(u_n)$ be an n-cube of X (a thin diagram with n cells).

For any $0 \leq i \leq n$, we say that the term u_i in c is *special* provided it is not an irreducible 1-path, that is, $\bar{u}_i \neq u_i$. For $1 \leq i \leq n$, we say that the term f_i in c is *special* provided $(u_{i-1}, f_i, *)$ is in T_l.

The n-cube c is called *essential* if it has no special terms. If c is not essential, then we find its leftmost special term. If this is one of the u_i's, then we call c *redundant*. Otherwise we call c *collapsible* (in this case the special term is one of the f_i's).

To describe the strict partial order $\succ$, we need to introduce one technical concept. Let $\Psi = \varepsilon(u_0) + g_1 + \cdots + g_k + \varepsilon(u_k)$ $(k \geq 0)$ be a thin diagram over $\mathcal{K}$. Suppose that for some i, j, where $0 \leq i \leq j \leq k$, we have $u_j = u'\lfloor g \rfloor u''$, for some 1-paths u', u'' and $g \in \mathbf{F}^-$. Then it is possible to *move the cell g_i to the right* replacing Ψ by a new thin diagram $\Psi' = \varepsilon(u_0) + \cdots + \varepsilon(u_{i-1}) + \varepsilon(\lfloor g_i \rfloor) + \cdots + \varepsilon(u') + g + \varepsilon(u'') + \cdots$ (we replace the term g_i by $\varepsilon(\lfloor g_i \rfloor)$ and the term $\varepsilon(u_j)$ by $\varepsilon(u') + g + \varepsilon(u'')$, thus we remove the cell g_i and insert a cell g). It is easy to see that this process of moving cells to the right always terminates. Indeed, Ψ' has also k cells and it can be represented in the form $\Psi' = \varepsilon(u_0') + g_1' + \cdots + g_k' + \varepsilon(u_k')$. If we compare the $(k+1)$-vectors $\mathbf{b} = (|u_k|, \ldots, |u_0|)$ and $\mathbf{b}' = (|u_k'|, \ldots, |u_0'|)$, then it follows from our description that $\mathbf{b}'$ strictly precedes $\mathbf{b}$ in the lexicographical order. Since these vectors have non-negative coordinates, the process of moving cells to the right must terminate.

Now we can define $\succ$. Let c, c' be redundant n-cubes of X. If $\lfloor c \rfloor \xrightarrow{+} \lfloor c' \rfloor$ then we set $c \succ c'$. Otherwise, if $\lfloor c \rfloor = \lfloor c' \rfloor$, then we set $c \succ c'$ whenever c' can be obtained from c by a (non-zero) number of moving cells to the right. The fact that $\mathcal{K}$ is Noetherian and the remark from the previous paragraph imply that $\succ$ is a strict partial order satisfying the descending chain condition.

Let $c = \varepsilon(u_0) + f_1 + \cdots + f_n + \varepsilon(u_n)$ be a redundant n-cube. This means that u_i is not irreducible for some $0 \leq i \leq n$ whereas all the terms to the left of u_i are not special. Let us find the edge (p, f, q) in T_l assigned to u_i. Thus $u_i = p\lfloor f \rfloor q$. Note that p is irreducible by definition. Thus the $(n+1)$-cube $\varepsilon(u_0) + f_1 + \cdots + f_i + \varepsilon(p) + f + \varepsilon(q) + \cdots$ is collapsible. We denote it by $\hat{c}$ and check that c is the free face of $\hat{c}$. We consider all n-faces $\lceil \hat{c} \rceil_j$ and $\lfloor \hat{c} \rfloor_j$ of $\hat{c}$, $1 \leq j \leq n+1$. Let $j = i+1$. Then $\lfloor \hat{c} \rfloor_j = c$ and $c' = \lceil \hat{c} \rceil_j$ is obtained from c by replacing $\lfloor f \rfloor$ by $\lceil f \rceil$. Since f participates in an edge from T_l, one has $\lfloor f \rfloor \xrightarrow{+} \lceil f \rceil$. Hence $c \succ c'$ whenever c' is redundant.

Now suppose that $j > i+1$. Then $\lfloor \hat{c} \rfloor_j$ and $\lceil \hat{c} \rceil_j$ are collapsible n-cubes. Thus we may skip this case because we compare c with redundant cubes only.

Finally, take $j \leq i$. We can only consider the case when $c' = \lfloor \hat{c} \rfloor_j$ since $\lfloor f_j \rfloor \xrightarrow{+} \lceil f_j \rceil$ or $\lfloor f_j \rfloor = \lceil f_j \rceil$. Suppose that c' is a redundant n-cube. To compare c and c', notice that their bottom paths are the same. The diagram c' is obtained by moving one cell of c to the right (the cell f_j has been deleted and the cell f has been added). Thus $c \succ c'$.

It remains to check that we have a bijection between redundant n-cubes and collapsible $(n+1)$-cubes. We already assigned a collapsible $(n+1)$-cube $\hat{c}$ to each redundant n-cube c. If we start with a collapsible $(n+1)$-cube, then we can find its leftmost special term. This is some $f \in \mathbf{F}^-$. The cube then has the form $\cdots + \varepsilon(p) + f + \varepsilon(q) + \cdots$, where (p, f, q) is in T_l. Replacing f by $\lfloor f \rfloor$ gives us a redundant cube c. It follows directly from definitions that the cube $\hat{c}$ assigned to c is exactly the collapsible $(n+1)$-cube we started with. This completes our proof that we have defined a collapsible scheme for X.

Summarizing and taking into account Lemma 7.1, we get the following.

Theorem 7.2. *Suppose that $\mathcal{K}$ is a complete directed 2-complex and let $G = \mathcal{D}(\mathcal{K}, w)$, where w is a non-empty 1-path in $\mathcal{K}$. Then there exists a $K(G,1)$ CW complex Y_w whose n-dimensional cells are in one-to-one correspondence with thin diagrams of the form $c = \varepsilon(u_0) + f_1 + \cdots + f_n + \varepsilon(u_n)$, where $n \geq 0$, the 1-paths u_i are irreducible for all $0 \leq i \leq n$, the edges $(u_{i-1}, f_i, *)$ are not in T_l for all $1 \leq i \leq n$, and $\lceil c \rceil$ is homotopic to w in $\mathcal{K}$.*

Note that each thin diagram c described in the statement of Theorem 7.2 is an essential cube in the Squier complex $\mathrm{Sq}(\mathcal{K})$.

Recall that a directed 2-complex is called 2-path connected if all non-empty 1-paths in it are homotopic. Let us call a directed 2-complex *almost 2-path connected* if the number of classes of homotopic 1-paths is finite (i. e., $\mathrm{Sq}(\mathcal{K})$ has finitely many connected components). A complex of the form $\mathcal{K}_{\mathcal{P}}$, where $\mathcal{P}$ is a semigroup presentation, is almost 2-path connected if and only if the semigroup given by $\mathcal{P}$ is finite. Thus the following statement generalizes a result from [14] and strengthens [16, Theorem 10.7].

Theorem 7.3. *Let $\mathcal{K}$ be a finite almost 2-path connected 2-complex. Then all diagram groups of $\mathcal{K}$ are of type $\mathcal{F}_\infty$.*

Proof. First suppose that $\mathcal{K}$ is complete. Let C denote the number of homotopy classes of 1-paths in $\mathcal{K}$ (the empty 1-paths are included) and let N be the number of positive 2-cells of $\mathcal{K}$. It is clear that the number of n-cells in the $K(G,1)$ space Y_w does not exceed $C(CN)^n$. In particular, it is finite so G has type $\mathcal{F}_\infty$.

Now suppose that $\mathcal{K}$ is not necessarily complete. By Lemma 6.1, $\mathcal{K}$ is contained in a finite complete almost 2-path connected directed 2-complex $\mathcal{K}'$, and the diagram groups of $\mathcal{K}$ are retracts of the diagram groups of $\mathcal{K}'$. It remains to recall that a retract of an $\mathcal{F}_\infty$ group is of type $\mathcal{F}_\infty$. □

By [16, Theorem 10.3], if $\mathcal{P}$ is a finite complete rewrite system such that all diagram groups over it are finitely generated, then all of them are finitely presented. Now we can deduce a much stronger result. In fact we can even eliminate the assumption that the presentation is finite.

Theorem 7.4. *Let $\mathcal{K}$ be a complete directed 2-complex. Suppose that all diagram groups of $\mathcal{K}$ are finitely generated. Then all of them are of type $\mathcal{F}_\infty$.*

Proof. By Theorem 7.2 it is enough to prove that for any $n \ge 1$, each connected component of $\mathrm{Sq}(\mathcal{K})$ has only finitely many essential n-cubes. We proceed by induction on n. Let $n = 1$. By definition, the set of essential cubes of dimension 1 is in one-to-one correspondence with the generating set of the corresponding diagram group described in Theorem 6.6. Since this set is minimal by Remark 6.7, it is finite.

Now let $n > 1$. For any essential 1-cube $\varepsilon(p) + f + \varepsilon(q)$ of a connected component $\mathrm{Sq}(\mathcal{K}, w)$ of $\mathrm{Sq}(\mathcal{K})$, let us consider the set of all essential $(n-1)$-cubes in $\mathrm{Sq}(\mathcal{K}, q)$. By the inductive assumption, it is finite. It remains to note that any essential n-cube $c = \varepsilon(u_0)+f_1+\cdots+f_n+\varepsilon(u_n)$ of $\mathrm{Sq}(\mathcal{K}, w)$ is determined uniquely by an essential 1-cube $\varepsilon(u_0) + f_1 + \varepsilon(q)$ of $\mathrm{Sq}(\mathcal{K}, w)$ and an essential $(n-1)$-cube $\varepsilon(u_1) + f_2 + \cdots + f_n + \varepsilon(u_n)$ from $\mathrm{Sq}(\mathcal{K}, q)$. □

Remark 7.5. Note that one can extract a stronger fact from the proof of Theorem 7.4. Suppose that $\mathcal{K}$ is a complete directed 2-complex. If some diagram group $\mathcal{D}(\mathcal{K}, w)$ is not of type $\mathcal{F}_\infty$, then there are two 1-paths $w'w_1$ and w_2w'' homotopic to w such that the diagram groups $\mathcal{D}(\mathcal{K}, w')$ and $\mathcal{D}(\mathcal{K}, w'')$ are not finitely generated.

Now we are going to prove that for any complete directed 2-complex $\mathcal{K}$, the complex Y_w described in the statement of Theorem 7.2 is in fact "minimal". Namely, for every $n \ge 0$, we shall compute the integer nth homology group of every diagram group $\mathcal{D}(\mathcal{K}, w)$ of $\mathcal{K}$ and show that it is a free Abelian group whose rank coincides with the number of n-cells in Y_w (i. e., the number of essential n-cubes in $\mathrm{Sq}(\mathcal{K}, w)$).

Let $G = \mathcal{D}(\mathcal{K}, w)$. Since the homology groups of a group G coincide with the homology groups of any $K(G,1)$ CW complex, let us consider the complex $X = \mathrm{Sq}(\mathcal{K}, w)$ (it is a $K(G,1)$ by Theorem 3.6). As usual, let T_l be a left forest in X.

Denote by P_n the free Abelian group with the set of n-cubes of X as a free basis. The boundary maps $\partial_n\colon P_n \to P_{n-1}$ ($n \geq 1$) are given by the formulas of Serre [29, p. 440] (see also [9]):

$$\partial_n(c) = \sum_{i=1}^{n} (-1)^i (\lceil c \rceil_i - \lfloor c \rfloor_i), \tag{14}$$

where c is an n-cube. Since the maps (14) form a chain complex [29], the nth integer homology group $H_n(G;\mathbb{Z})$ coincides with the nth homology group of that chain complex (i. e., $\operatorname{Ker}\partial_n/\operatorname{Im}\partial_{n+1}$).

As in [8, 12], we define an endomorphism ϕ of the chain complex $P = (P_n, \partial_n)$. This endomorphism maps every P_n into the subgroup Q_n of P_n freely generated by the essential n-cubes. Let c be an n-cube (a generator of P_n). If c is collapsible then we set $\phi(c) = 0$. If c is essential then we set $\phi(c) = c$. Finally suppose that c is redundant. In that case we proceed by the Noetherian induction on $\succ$.

Since c is redundant, there exists a collapsible $(n+1)$-cube $\hat{c}$ such that c is the free face of $\hat{c}$. Then $\partial_{n+1}(\hat{c}) = \pm c + \Sigma$ for some linear combination Σ of cubes that are either essential or collapsible or redundant but smaller than c with respect to $\succ$. Thus we can assume that $\phi(c')$ has been defined already for all c' occurring in Σ. So we can set $\phi(c) = \mp\phi(\Sigma)$.

It is shown in [12] (see also [8, p. 150]), that ϕ indeed is an endomorphism of the chain complex (that is, it commutes with the boundary maps), and that the chain complex Q formed by the groups Q_n and boundary maps $\delta_n = \phi\partial_n$ is chain-equivalent to the initial chain complex. Thus the homology groups of Q coincide with the homology groups of P.

We are going to prove that δ_n is a zero map. For that we need the following statement.

Lemma 7.6. *For any n-cube $c = \varepsilon(u_0) + f_1 + \cdots + f_n + \varepsilon(u_n)$ of $\operatorname{Sq}(\mathcal{K}, w)$, we let $\bar{c} = \varepsilon(\bar{u}_0) + f_1 + \cdots + f_n + \varepsilon(\bar{u}_n)$. Then $\phi(c) = \bar{c}$ if $\bar{c}$ is essential and $\phi(c) = 0$ if $\bar{c}$ is collapsible (note that $\bar{c}$ cannot be redundant by definition).*

Proof. Note that if c is essential then $c = \bar{c}$ and $\phi(c) = c$ as required.

Suppose that c is collapsible. Then $\phi(c) = 0$ by definition. Thus we only need to check that $\bar{c}$ is also collapsible. Since c is collapsible, the edge $(u_{i-1}, f_i, *)$ is in the left forest T_l for some $i \leq n$ and all the u_j's are irreducible for $0 \leq j < i$. Then $\bar{c}$ possesses similar properties whence $\bar{c}$ is collapsible.

Now suppose that c is redundant. Take the smallest number $i \leq n$ such that u_i is not irreducible, and consider the edge (p, f, q) from T_l assigned to u_i. This i will be called the *index* of c. Consider also the collapsible $(n+1)$-cube $\hat{c} = \varepsilon(u_0) + \cdots + f_i + \varepsilon(p) + f + \varepsilon(q) + \cdots$ whose free cell is c. Since $\mathcal{K}$ is Noetherian, we can assume without loss of generality that the statement of the lemma does not hold for c but holds for all n-cubes c' such that $\lfloor c \rfloor \xrightarrow{+} \lfloor c' \rfloor$. We can also assume that among all such counterexamples, c has the smallest index i.

Notice that for every j, $1 \leq j \leq n+1$,

$$\overline{\lceil \hat{c} \rceil_j} = \overline{\lfloor \hat{c} \rfloor_j}. \tag{15}$$

Also notice that $c = \lfloor \hat{c} \rfloor_{i+1}$ and $\lfloor c \rfloor \xrightarrow{+} \lfloor c' \rfloor$, where $c' = \lceil \hat{c} \rceil_{i+1}$. By (15), $\bar{c} = \overline{c'}$. By the definition of ϕ, we have $\phi(c) = \phi(c') + \phi(\Sigma)$, where Σ is the sum of $\lceil \hat{c} \rceil_j - \lfloor \hat{c} \rfloor_j$, $j \neq i+1$ by (14).

It remains to check that

$$\phi(\lceil \hat{c} \rceil_j - \lfloor \hat{c} \rfloor_j) = 0 \tag{16}$$

for every $j \neq i+1$. If $j > i+1$, then both cells $\lceil \hat{c} \rceil_j$ and $\lfloor \hat{c} \rfloor_j$ are collapsible, so $\phi(\lceil \hat{c} \rceil_j) = \phi(\lfloor \hat{c} \rfloor_j) = 0$ and (16) holds.

Let $1 \leq j \leq i$. The thin diagrams $e' = \lceil \hat{c} \rceil_j$ and $e = \lfloor \hat{c} \rfloor_j$ are obtained from $\hat{c}$ by replacing f_j by its top and bottom path, respectively. Suppose that $e' \neq e$ (otherwise there is nothing to prove). Thus $\lfloor f_j \rfloor \xrightarrow{+} \lceil f_j \rceil$ and the statement of the lemma holds for e' because $\lfloor c \rfloor = \lfloor e \rfloor \xrightarrow{+} \lfloor e' \rfloor$. It is easy to see from definition that the cube e is redundant. The index of e is $j-1 < i$. Hence the statement of the lemma holds for e as well. By (15), $\overline{e'} = \bar{e}$. Therefore, $\phi(e'-e) = 0$ and (16) holds. □

Now it is easy to show that all boundary maps δ_n, $n \geq 1$, in the chain complex Q are zero. Indeed, by Lemma 7.6, the value $\phi(c)$ depends only on $\bar{c}$. By (15) and (14), for every n-cube c,

$$\delta_n(c) = \phi\partial_n(c) = \phi\left(\sum_{i=1}^{n}(-1)^i(\lceil c \rceil_i - \lfloor c \rfloor_i)\right) = \sum_{i=1}^{n}(-1)^i(\phi(\lceil c \rceil_i) - \phi(\lfloor c \rfloor_i)) = 0.$$

Thus $\operatorname{Ker}\partial_n = Q_n$ and $\operatorname{Im}\partial_{n+1} = 0$. Hence for $H_n(G;\mathbb{Z}) \cong Q_n$ is free Abelian, $n \geq 1$.

If $n = 0$, then $H_0(G;\mathbb{Z}) \cong \mathbb{Z}$ and we have only one essential cell of dimension 0 — this is the vertex corresponding to the irreducible 1-path of $\mathrm{Sq}(\mathcal{K}, w)$. Thus we proved

Theorem 7.7. *Let $\mathcal{K}$ be a complete directed 2-complex and let w be a non-empty 1-path in $\mathcal{K}$. The nth integer homology $H_n(G;\mathbb{Z})$ ($n \geq 0$) of the diagram group $G = \mathcal{D}(\mathcal{K}, w)$ is free Abelian. Its free basis consists of all essential n-cubes from $\mathrm{Sq}(\mathcal{K}, w)$.*

Theorem 7.7 implies that the CW complex Y_w from Theorem 7.2 gives a minimal presentation of $G = \mathcal{D}(\mathcal{K}, w)$ in terms both the number of generators and the number of relations. In fact, it gives a minimal set of generators of homology groups in all dimensions.

Remark 7.8. It is not difficult to prove that the presentation given by Y_w is precisely the presentation from Theorem 6.6, where T_r is replaced by T_l (see Remark 6.7, part b). We already know (Remark 6.7, part a) that the presentation from Theorem 6.6 involves the minimal possible number of generators. Let us show that it contains the minimal number of relations as well. For any 1-path p, let us denote by $\mu(p)$ the minimal number of generators for the diagram group $G = \mathcal{D}(\mathcal{K}, w)$. This is exactly the number of edges of the form $(u, f, v) \notin T_l$ that belong to $\mathrm{Sq}(\mathcal{K}, w)$, where u, v are irreducible, $f \in F^-$ (these are the essential 1-cubes of $\mathrm{Sq}(\mathcal{K}, w)$). If we replace here T_l by T_r, then we again have a minimal generating set of G because of a symmetry. It is easy to give a formula to compute the number of the defining relations of G given by Theorem 7.2. Let $s_1, \ldots, s_m$ be the third components of the essential 1-cubes of G ($m = \mu(w)$). Each of the defining relations corresponds to an essential 1-cube in $\mathrm{Sq}(\mathcal{K}, s_i)$ for some i. Thus the sum $\mu(s_1) + \cdots + \mu(s_m)$ is exactly the number of the defining relations of G given by Theorem 7.2. Clearly, it will be the same if we use T_r instead of T_l in the definition of essential cubes. Thus the number of defining relations given by Theorem 6.6 and Theorem 7.2 are the same.

Now consider the homology groups of arbitrary diagram groups. Let H be a diagram group of an arbitrary directed 2-complex. We know from Lemma 5.1, part 4 that H is a retract of a diagram group G of a complete directed 2-complex. Notice that the retraction can be described in the language of group homomorphisms: H is a retract of G if and only if there are two homomorphisms $\phi\colon G \to H$ and $\psi\colon H \to G$ such that $\phi\psi = \mathbf{id}_H$ (ψ acts first). Since $H_n(-,\mathbb{Z})$ is a covariant functor [7], this implies that $H_n(H;\mathbb{Z})$ is a retract of $H_n(G;\mathbb{Z})$. In particular, $H_n(H;\mathbb{Z})$ is also free Abelian and its rank does not exceed the rank of $H_n(G;\mathbb{Z})$. So we proved

Theorem 7.9. *For any $n \geq 0$ and for any diagram group G, the nth integer homology group $H_n(G;\mathbb{Z})$ is free Abelian.*

If G is a group of type $\mathcal{F}_\infty$, then one can consider its Poincaré series

$$P_G(t) = \sum_{n=0}^{\infty} r_n t^n,$$

where r_n denotes the rank of the nth integer homology group of G. Note that $r_0 = 1$.

Example 7.10. a) Let $\mathcal{K} = \langle\, x \mid x^r = x \,\rangle$ $(r \geq 2)$. It is proved in [16], that the diagram group $F_r = \mathcal{D}(\mathcal{K}, x)$ is the generalization of the R. Thompson group F defined in [8]. It is proved in [16] that $\mathcal{D}(\mathcal{K}, x) \cong F_r$, where F_r is a generalization of the Thompson group $F = F_2$ defined in [8]. That complex is complete so we can use Theorems 7.2 and 7.7. The essential n-cells of this complex $(n \geq 1)$ have the form $c = \varepsilon(x^{k_0}) + (x = x^r) + \varepsilon(x^{k_1}) + \cdots + (x = x^r) + \varepsilon(x^{k_n})$, where $1 \leq k_i < r$ for $0 \leq i < n$, $0 \leq k_n < r$. These cells may belong to different components of $\mathrm{Sq}(\mathcal{K})$. Clearly, c belongs to the component of x if and only if the sum $n + k_0 + k_1 + \cdots + k_n$ equals 1 modulo $r - 1$. This condition determines uniquely the number k_0 by the other numbers. So there are exactly $r \cdot (r-1)^{n-1}$ ways to choose c in $\mathrm{Sq}(\mathcal{K}, x)$. (For instance, in the case $r = 2$ we have the R. Thompson's group F, which has $\mathbb{Z}^2$ as its nth integer homology group for $n \geq 1$, which was proved in [9].)

The Poincaré series for F_r has the form

$$P(t) = 1 + rt + r(r-1)t^2 + \cdots + r(r-1)^{n-1}t^n + \cdots = \frac{1+t}{1-(r-1)t}.$$

b) Now let $\mathcal{V} = \langle\, y \mid y^3 = y^2 \,\rangle$ be the complex on Figure 9 (by Theorem 5.7, the diagram group $\mathcal{D}(\mathcal{V}, y^2)$ is universal). The complex $\mathcal{V}$ is complete as well. The essential n-cubes in $\mathrm{Sq}(\mathcal{V}, x^2)$ have the form $c = \varepsilon(y^{k_0}) + (y^2 = y^3) + y^{k_1} + \cdots + (y^2 = y^3) + \varepsilon(y^{k_n})$, where $k_i = 1, 2$ for $0 \leq i < n$, $k_n = 1, 2, 3$. All of them for $n \geq 1$ belong to $\mathrm{Sq}(\mathcal{K}, y^2)$. Hence the rank of the nth homology group of $\mathcal{D}(\mathcal{V}, x^2)$ is $3 \cdot 2^n$ $(n \geq 1)$ and so the Poincaré series is

$$P(t) = 1 + 6t + 12t^2 + \cdots + 3 \cdot 2^n t^n + \cdots = \frac{1+4t}{1-2t}.$$

Notice that the Poincaré series of $\mathcal{D}(\mathcal{V}, x^2)$ coincides with the Poincaré series of the free product $F_3 * F_3$ but these diagram groups are not isomorphic (which can be proved by using Kurosh's theorem).

c) One more universal diagram group is given by the complete directed 2-complex $\mathcal{H}_1 = \langle\, x \mid x^2 = x, x = x \,\rangle$ (Theorem 5.6). Let us find the number of the essential n-cubes in $\mathrm{Sq}(\mathcal{H}_1, x)$. Let $c = \varepsilon(u_0) + f_1 + \cdots + f_n + \varepsilon(u_n)$ be one of these cubes. Then there are three possibilities for each of the pairs (u_{i-1}, f_i), namely, $(x, x = x^2)$, $(1, x = x)$, or $(x, x = x)$ $(1 \leq i \leq n)$. There are two possibilities for the 1-path u_n, namely, 1 and x. So the rank of the nth homology group of $\mathcal{G}_1 = \mathcal{D}(\mathcal{H}_1, x)$ equals $2 \cdot 3^n$ for $n \geq 1$. It is interesting to mention that both universal groups considered in b), c) have the same minimal number of generators (equal to 6) and we know that they are embeddable into each other because of their universal property. However, they are not isomorphic because their second homology groups have ranks 12 and 18, respectively. Thus the Poincaré series of $\mathcal{G}_1$ is

$$P(t) = 1 + 6t + 18t^2 + \cdots + 2 \cdot 3^n t^n + \cdots = \frac{1+3t}{1-3t}.$$

We see that all these Poincaré series are rational. This can be explained by the following

Theorem 7.11. *Let $\mathcal{K}$ be a complete finite almost 2-path connected directed 2-complex. Then the Poincaré series of any of its diagram group is rational.*

Proof. We refer to [24] for the well-known properties of rational languages. Let $A = \mathbf{P} \cup (\mathbf{P} \times \mathbf{F}^-)$, where $\mathbf{P}$ consists of all irreducible 1-paths in $\mathcal{K}$, including the empty 1-paths, $\mathbf{F}^-$ consists of all negative 2-cells of $\mathcal{K}$.

Let 0 be a symbol not in $\mathbf{P}$, and let us define a binary operation $\cdot$ on $S = \mathbf{P} \cup \{0\}$: $p \cdot q = \overline{pq}$ if $\tau(p) = \iota(q)$, $p, q \in \mathbf{P}$; all other products are equal to 0. It is easy to see that S is a semigroup.

Let ϕ be a homomorphism from the free semigroup A^+ to S induced by the map that takes each $p \in \mathbf{P}$ to itself and each pair $(p, f) \in \mathbf{P} \times \mathbf{F}^-$ to $\overline{p\lceil f \rceil} = \overline{p\lfloor f \rfloor}$. Notice that for every $s \in \mathbf{P}$, every word w of the form $(u_0, f_1) \cdots (u_{n-1}, f_n)u_n$ from the rational language $\phi^{-1}(s) \subseteq A^+$ corresponds to an n-cube in $\mathrm{Sq}(\mathcal{K}, s)$. This cube is essential if and only if w does not contain letters of the form (p, f), where $p \in \mathbf{P}$, $f \in \mathbf{F}^-$, and $(p, f, *)$ belongs to the fixed left forest T_l of $\mathrm{Sq}(\mathcal{K})$.

Thus let L be the sublanguage of $\phi^{-1}(s)$ consisting of all words from $\phi^{-1}(s) \cap A^+\mathbf{P}$ which do not contain the letters described in the previous paragraph. Clearly, L is a rational language. There exists a one-to-one correspondence between words of length $n+1$ in L and essential n-cubes in $\mathrm{Sq}(\mathcal{K}, s)$. Since the generating function of a rational language L is rational (see for example [11]), the Poincaré series of $\mathrm{Sq}(\mathcal{K}, s)$ is a rational function as well. □

Recall [7] that for any group G, its geometric dimension, $\mathrm{gd}(G)$, is the smallest dimension of a $K(G, 1)$. Its cohomological dimension, $\mathrm{cd}(G)$, is the length of the shortest projective resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. It is easy to see [7] that

$$\mathrm{cd}(G) \leq \mathrm{gd}(G). \tag{17}$$

By the Eilenberg – Ganea theorem [23], $\mathrm{cd}(G) = \mathrm{gd}(G)$ provided $\mathrm{cd}(G) \neq 2$ or $\mathrm{gd}(G) \neq 3$.

Theorems 7.2 and 7.7 immediately imply that for diagram groups over complete directed 2-complexes these two dimensions coincide.

Theorem 7.12. *For every diagram group G over a complete directed 2-complex,*

$$\mathrm{cd}(G) = \mathrm{gd}(G).$$

Proof. Let $G = \mathcal{D}(\mathcal{K}, w)$. By Theorem 7.7 the length of any projective resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ cannot be smaller than the highest dimension n of an essential cube of $\mathrm{Sq}(\mathcal{K}, w)$. By Theorem 7.2, $n \geq \mathrm{gd}(G)$. Therefore, $\mathrm{cd}(G) \geq n \geq \mathrm{gd}(G)$. Hence by (17), $\mathrm{gd}(G) = \mathrm{cd}(G)$. □

The following result gives an algebraic characterization of groups of finite cohomological dimension among diagram group of complete directed 2-complexes.

Theorem 7.13. *Let G be a diagram group over a complete directed 2-complex $\mathcal{K}$, and n be a natural number. Then $\mathrm{cd}(G) \geq n$ if and only if G contains a copy of $\mathbb{Z}^n$.*

Proof. Let $G = \mathcal{D}(\mathcal{K}, w)$. Let T_l be a left forest in $\mathrm{Sq}(\mathcal{K})$. The "if" statement is well known [7]. So suppose that $\mathrm{cd}(G) \geq n$. Then by Theorem 7.7, $\mathrm{Sq}(\mathcal{K}, w)$ contains an essential cube $c = \varepsilon(u_0) + f_1 + \cdots + f_n + \varepsilon(u_n)$.

By definition, for every $1 \leq i \leq n$, the edge $(u_{i-1}, f_i, *)$ does not belong to T_l. Hence the connected component $\mathrm{Sq}(\mathcal{K}, u_{i-1}\lfloor f_i \rfloor)$ contains edges not in T_l. Therefore, by Theorem 6.6

(or Theorem 7.7), the group $G_i = \mathcal{D}(\mathcal{K}, u_{i-1}\lfloor f_i \rfloor)$ is non-trivial. Since all diagram groups are torsion-free [16], G_i contains a copy of $\mathbb{Z}$.

By Corollary 3.5 the diagram group $G_1 \times \cdots \times G_n$ embeds into $H = \mathcal{D}(\mathcal{K}, p)$, where $p = u_0\lfloor f_1 \rfloor \cdots \lfloor f_n \rfloor u_n$. Therefore, a copy of $\mathbb{Z}^n$ is contained in H. But by the choice of c, the 1-paths p and w are in the same connected component of $\mathrm{Sq}(\mathcal{K})$. Hence by Corollary 3.4, H is isomorphic to G, so G contains a copy of $\mathbb{Z}^n$, as required. □

Notice that Theorem 8.4 below gives a characterization of directed 2-complexes $\mathcal{K}$ such that $\mathcal{D}(\mathcal{K}, w)$ contains a copy of $\mathbb{Z}^n$.

Theorem 7.13 immediately implies the following result.

Theorem 7.14. *A diagram group G over a complete directed 2-complex is free if and only if G does not contain a copy of $\mathbb{Z}^2$. In particular, a hyperbolic group can be a diagram group of a complete directed 2-complex if and only if it is free.*

Proof. Indeed, by Theorem 7.12, if G does not contain $\mathbb{Z}^2$ then $\mathrm{cd}(G) \leq 1$, and one can use the well known result of Stallings and Swan [23] (or, easier, one can use Remark 7.8, and conclude that G has a presentation with no relations). □

Problem 7.15. *Is it possible to drop the completeness restriction from the formulations of Theorems 7.11, 7.12, 7.13, 7.14?*

Remark 7.16. Recall that in Section 3, we identified $\mathrm{Sq}(\mathcal{K})$ with the space of positive paths Ω_+ in the directed 2-complex $\mathcal{K}$. By Theorem 3.6, the homology of the connected components of that space coincide with the homology of the corresponding diagram group. Hence by Theorem 7.11, the Poincaré series of the space of positive paths of a complete almost 2-path connected directed 2-complex is rational. This resembles the well known result of Serre (see, for example, [1]) that the Poincaré series of the loop space of a simply connected CW 2-complex is always rational.

Remark 7.17. Notice that the completeness restriction in the statements of this paper can be replaced by the condition "there exists a left forest". Say, let $\mathcal{K} = \langle\, a, b \mid ab = a, ba = b \,\rangle$. It is not hard to check that $\mathcal{K}$ is not complete. However, one can construct a spanning forest satisfying conditions (F1) and (F2) of the left forest (it is formed by all edges of the form $(1, a = ab, *)$, $(a, b = ba, *)$, $(1, b = ba, *)$, $(b, a = ab, *)$ and the inverse edges). Using that forest, as above, one can compute the presentation of the corresponding diagram groups, and their homology groups. The Poincaré series of the diagram groups of this complex are rational.

8 Rigidity

Recall that the flat torus theorem [3, Theorem 7.1] says, in particular, that if X is a metric space with CAT(0) universal cover $\tilde{X}$ and the fundamental group of X contains a copy of $\mathbb{Z}^n$, then X contains a π_1-embedded torus $\mathbb{R}^n/\mathbb{Z}^n$.

Results of this section are of similar spirit. They say that a diagram groupoid of a directed 2-complex $\mathcal{K}$ contains certain diagram group $G = \mathcal{D}(\mathcal{S}, p)$ if and only if there exists a p-nonsingular morphism from $\mathcal{S}$ into $\mathcal{K}$. Since every p-nonsingular morphism $\phi\colon \mathcal{S} \to \mathcal{K}$ induces a π_1-injective continuous map $\mathrm{Sq}(\mathcal{S}, p) \to \mathrm{Sq}(\mathcal{K}, \phi(p))$, these results (and Theorem 3.3) imply that if $\pi_1(\mathrm{Sq}(\mathcal{K}), u)$ contains a copy of $G = \pi_1(\mathrm{Sq}(\mathcal{S}), p)$ then there exists a π_1-injective continuous map from $\mathrm{Sq}(\mathcal{S}, w)$ into $\mathrm{Sq}(\mathcal{K})$.

In general we say that a triple (diagram group G, directed 2-complex $\mathcal{K}$, 1-path p in $\mathcal{K}$) is *rigid* if $G = \mathcal{D}(\mathcal{K}, p)$ and for every directed complex $\mathcal{K}'$ such that $\mathcal{D}(\mathcal{K}')$ contains a copy of G there exists a p-nonsingular morphism of $\mathcal{K}$ into $\mathcal{K}'$.

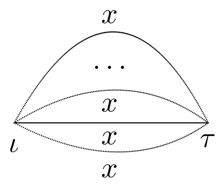

Figure 13.

For example, consider the directed 2-complex $\mathcal{K}$ on Figure 13 with two vertices ι and τ, one edge x connecting ι with τ and positive 2-cells of the form $x = x$ labelled by elements of some set A. By Theorem 6.6 (or a straightforward computation), the diagram group $G = D(\mathcal{K}, x)$ is the free group of rank $|A|$. It is easy to see (exercise) that the triple $(G, \mathcal{K}, x)$ is rigid.

A much more nontrivial example of a rigid triple involves the R. Thompson group F. Example 6.8 shows that F is the diagram group of the Dunce hat, $\mathcal{H}_0$.

The following proposition shows a remarkable property of the Dunce hat.

Theorem 8.1. *Every morphism from the Dunce hat to any directed 2-complex $\mathcal{K}$ is nonsingular.*

Proof. Let us consider any morphism ϕ from $\mathcal{H}_0$ into a directed 2-complex $\mathcal{K}$. We need to show that it is p-nonsingular for every non-empty 1-path p in $\mathcal{H}_0$. Since for every such p, the diagram groups $\mathcal{D}(\mathcal{H}_0, p)$ and $\mathcal{D}(\mathcal{H}_0, x^5)$ are conjugate, it is enough to show that ϕ is x^5-nonsingular. Since x^5 and x are homotopic, we have $\mathcal{D}(\mathcal{H}_0, x^5) \cong \mathcal{D}(\mathcal{H}_0, x) \cong F$ (see Example 6.8).

Since all proper homomorphic images of the group F are Abelian [13], it suffices to find an element in the derived subgroup of $\mathcal{D}(\mathcal{H}_0, x^5)$ that is not in the kernel of ϕ_{x^5}. Let Π be the diagram over $\mathcal{H}_0$ corresponding to the atomic 2-path $(1, f, 1)$, where f is the positive 2-cell $x^2 = x$ in $\mathcal{H}_0$. Consider the diagram $\Psi = \varepsilon(x) + \Pi + \Pi^{-1} + \varepsilon(x)$ from $\mathcal{D}(\mathcal{H}_0, x^5)$. By Theorem 11.3 from [16], Ψ is in the derived subgroup of $\mathcal{D}(\mathcal{H}_0, x^5)$ and is nontrivial. By Theorem 3.3, we can assume that the diagram $\Delta = \phi_{x^5}(\Pi)$ is reduced. Since it is a $(\phi(x)^2, \phi(x))$-diagram, it is nontrivial. Then $\phi_{x^5}(\Psi) = \varepsilon(\phi(x)) + \Delta + \Delta^{-1} + \varepsilon(\phi(x))$ is also a reduced and nontrivial diagram, hence by Theorem 3.3, $\phi_{x^5}(\Psi) \neq 1$. □

Note that the same property is true for directed 2-complexes $\langle\, x \mid x^r = x \,\rangle$ that correspond to groups F_r $(r \geq 2)$, the generalizations of F (see [6, 16]). The proof is based on the same idea (all proper homomorphic images of these groups are also Abelian).

Here is a reformulation of the main result of [18] which shows that the triple $(F, \mathcal{H}_0, x)$ is rigid.

Theorem 8.2. ([18]) *Let $\mathcal{K}$ be a directed complex. Then the following conditions are equivalent.*

1. *A diagram groupoid $\mathcal{K}$ contains an isomorphic copy of the R. Thompson group F.*
2. *The complex $\mathcal{K}$ contains a non-empty 1-path which is homotopic to its square.*
3. *There exists a (nonsingular) morphism from the Dunce hat to $\mathcal{K}$.*

Thus if the diagram groupoid of $\mathcal{K}$ contains a copy of F then it contains a naturally embedded copy of F.

Another example of a rigid triple is given by [17, Theorem 24]. Let $\mathcal{Q}$ be the directed 2-complex with three vertices, three edges x, y, z and three positive cells of the forms $xy = x$, $y = y$, $yz = z$ on Figure 14 (to obtain the complex from the diagram , we identify all edges having the same labels).

It is proved in [17] that the diagram group $\mathcal{D}(\mathcal{Q}, xyz)$ is isomorphic to the restricted wreath product $\mathbb{Z}$ wr $\mathbb{Z}$. Theorem 24 of [17] shows that the triple ($\mathbb{Z}$ wr $\mathbb{Z}$, $\mathcal{Q}, xyz$) is rigid.

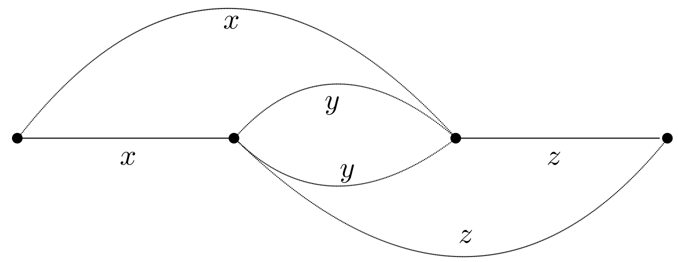

Figure 14.

The free Abelian group $\mathbb{Z}^n$ can participate in a rigid triple too (for every $n \geq 1$). In fact, using a description of commuting diagrams ([17, Theorem 17]) one can obtain a much more precise result (Theorem 8.4 below).

Let $\mathcal{S}_n$ be the following directed 2-complex: take a simple path labelled by the word $w_n = x_1 \cdots x_n$, where x_i are letters and let us attach n positive 2-cells $x_1 = x_1$, ..., $x_n = x_n$ to it. Thus $\mathcal{S}_n$ is a chain of n spheres (Figure 15).

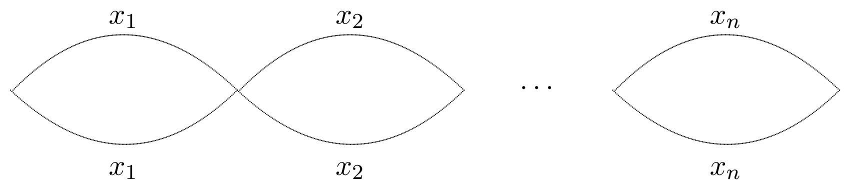

Figure 15.

Then $\mathrm{Sq}(\mathcal{S}_n, w_n)$ is the n-dimensional torus (this is easy to check). Thus by Theorem 3.3, the group $\mathcal{D}(\mathcal{S}_n, w_n)$ is isomorphic to $\mathbb{Z}^n$.

The next result is one of the most useful technical facts about diagram groups. In [17], it is formulated and proved for diagrams over semigroup presentations (see [17, Theorem 24]). The proof for directed complexes is completely similar (in fact it can be deduced from the result of [17] by using subdivisions of complexes). It is similar to the well known theorem that commuting matrices over an algebraically closed field are simultaneously conjugate to their Jordan forms.

Lemma 8.3. *Let $\mathcal{K}$ be a directed 2-complex and w be a non-empty 1-path in $\mathcal{K}$. Suppose that A_1, ..., A_n are spherical (w, w)-diagrams that pairwise commute in G. Then there exist a 1-path $v = v_1 \dots v_m$, spherical (v_j, v_j)-diagrams Δ_j $(1 \leq j \leq m)$ over $\mathcal{K}$, integers d_{ij} $(1 \leq i \leq n$, $1 \leq j \leq m)$ and some (w, v)-diagram Γ over $\mathcal{K}$ such that*

$$\Gamma^{-1} A_i \Gamma = \Delta_1^{d_{i1}} + \cdots + \Delta_m^{d_{im}}$$

for all $1 \leq i \leq n$.

Theorem 8.4. *The triple* $(\mathbb{Z}^n, \mathcal{S}_n, w_n)$ *is rigid for every* $n \geq 1$. *In addition, let* $\mathcal{K}$ *be a directed 2-complex. Then every copy of* $\mathbb{Z}^n$ *in* $\mathcal{D}(\mathcal{K})$ *is conjugate in* $\mathcal{D}(\mathcal{K})$ *to a subgroup of a naturally embedded copy of* $\mathbb{Z}^m$ *for some* $m \geq n$.

Proof. Suppose that a diagram group $\mathcal{D}(\mathcal{K}, p)$ of some directed 2-complex contains a copy $G = \langle A_1, \ldots, A_n \rangle$ of $\mathbb{Z}^n$. We use the notation from Lemma 8.3. Let us also assume that m is chosen to be minimal. Then all diagrams $\Delta_1, \ldots, \Delta_m$ are nontrivial. Indeed, if we assume the contrary, then $m > 1$ because $G \neq 1$. If Δ_i is trivial for some i, then one has $i < m$ or $i > 1$. Without loss of generality we assume that $i < m$. But now it would be possible to replace Δ_i by Δ_{i+1} taking into account that the power of a sum is the sum of powers and all powers of a trivial diagram coincide.

Clearly, $m \geq n$ (otherwise the rank of the subgroup generated by $A_1, \ldots, A_n$ would be less than n).

Now the map ϕ from $\mathcal{S}_m$ to $\mathcal{K}$ that sends the positive 2-cell $x_i = x_i$ to the 2-path corresponding to Δ_i $(1 \leq i \leq m)$, defines a morphism. The image of $\mathcal{D}(\mathcal{S}_m, w_m)$ under ϕ_{w_n} is generated by the diagrams $\varepsilon(v_1 \cdots v_{i-1}) + \Delta_i + \varepsilon(v_{i+1} \cdots v_m)$. Thus the image of ϕ_{w_m} is isomorphic to $\mathbb{Z}^m$. Therefore, ϕ is w_m-nonsingular. Clearly, the naturally embedded copy $\phi(\mathcal{D}(\mathcal{S}_m, w_m))$ of $\mathbb{Z}^m$ contains $\Gamma^{-1}G\Gamma$. This proves the second statement of the theorem.

In order to prove the rigidity statement, we just note that $\mathcal{S}_n$ is a subcomplex of $\mathcal{S}_m$ so it maps into it nonsingularly. But we already have a nonsingular morphism of $\mathcal{S}_m$ into $\mathcal{K}$. It suffices to compose the morphisms. □

Problem 8.5. It is interesting to characterize other diagram groups that can participate in rigid triples. In particular, in view of rigidity of the triple $(F, \mathcal{H}_0, x)$ it is natural to ask if the analog is true for $\mathcal{H}_n$. By Theorem 5.6, it is enough to prove that for $n = 1$. This would give a characterization of universal directed 2-complexes as those admitting a nonsingular morphism from $\mathcal{H}_1$.

9 Diagram groups and group theoretic constructions

We have several results in [16, 17], which show that the class of diagram groups is closed under various group-theoretical constructions (free products, finite direct products, etc.). Directed 2-complexes are very convenient to illustrate these results. Basically all these constructions appear when we glue directed complexes with given diagram groups in a certain way.

The easiest way to glue two directed 2-complexes is the following. Let $\mathcal{K}_1$, $\mathcal{K}_2$ be disjoint directed 2-complexes with distinguished 1-paths p_1, p_2 in them, respectively. Let $\mathcal{K}$ be the union of $\mathcal{K}_1$ and $\mathcal{K}_2$ with identified vertices $\tau(p_1)$ and $\iota(p_2)$, and let $p = p_1p_2$. It follows from [16, Lemma 8.1] that $\mathcal{D}(\mathcal{K}, p)$ is isomorphic to the direct product of $\mathcal{D}(\mathcal{K}_1, p_1)$ and $\mathcal{D}(\mathcal{K}_2, p_2)$. Indeed, it is easy to see that $\mathrm{Sq}(\mathcal{K})$ is homeomorphic to the direct product of $\mathrm{Sq}(\mathcal{K}_1)$ and $\mathrm{Sq}(\mathcal{K}_2)$. More generally, if we have finitely many disjoint directed 2-complexes $\mathcal{K}_i$ $(1 \leq i \leq n)$ with distinguished 1-paths p_i in them, then we can take a union of these complexes and identify their vertices in such a way that $p = p_1 \cdots p_n$ becomes a path. We get a new directed 2-complex $\mathcal{K}$. The diagram group $G = \mathcal{D}(\mathcal{K}, p)$ is isomorphic to the direct product of the groups $G_i = \mathcal{D}(\mathcal{K}_i, p_i)$ $(1 \leq i \leq n)$.

Another way of gluing 2-complexes leads to the free product. Consider the following complex which we shall call a *switch*. It has 4 vertices, edges a, b, c, s and one positive 2-cell $a = bsc$ (see Figure 16).

For every directed 2-complex $\mathcal{K}$ with a distinguished non-empty 1-path p, we attach the switch to $\mathcal{K}$ by subdividing the edge s into $|p|$ edges and gluing s with p. Let $\mathcal{K}_a$ be the resulting directed 2-complex. It easy follows from the part 1) of Theorem 4.1 and from Corollary 3.4 that $\mathcal{D}(\mathcal{K}_a, a)$ and $\mathcal{D}(\mathcal{K}, p)$ are isomorphic.

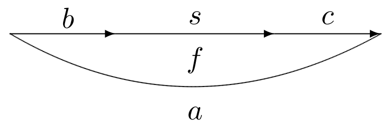

Figure 16.

Now take any collection of disjoint directed 2-complexes $\mathcal{K}_i$, $i \in I$, with distinguished 1-paths p_i, attach to the paths p_i disjoint copies of the switch with edges a_i, b_i, c_i, s_i, take the union of the resulting complexes, then glue all edges a_i together, and denote the common edge by a, and the resulting complex by $\mathcal{K}$.

Then the diagram group $G = \mathcal{D}(\mathcal{K}, a)$ is isomorphic to the free product of the family of groups $G_i = \mathcal{D}(\mathcal{K}_i, a_i)$ $(i \in I)$. Indeed, the the complex $\mathrm{Sq}(\mathcal{K}, a)$ is just the wedge of complexes $\mathrm{Sq}(\mathcal{K}_i, a_i)$, $i \in I$ (see [16, Theorem 8.4] and [17, Example 6]).

All these and some other useful constructions are particular cases of the concept of the diagram product of groups, see [17, Section 2].

Algebraically the diagram product is defined as follows (we translate the definition from [17] into the language of directed 2-complexes). Let $\mathcal{K}$ be a directed 2-complex, and let $\mathcal{G}_X = \{G_x \mid x \in X\}$ be a collection of groups then the diagram product $\mathcal{D}(\mathcal{G}_X; \mathcal{K}, p)$ of $\mathcal{G}_X$ over $\mathcal{K}$ is the fundamental group of the complex of groups [3] where

- the underlying 2-complex is the Squier complex $\mathrm{Sq}(\mathcal{K})$,
- the vertex group $G(s)$ assigned to a vertex $s = x_1 \cdots x_n$, $x_i \in X$, is the direct product $G_{x_1} \times \cdots \times G_{x_n}$,
- the edge group assigned to the edge (p, f, q) is the direct product $G(p) \times G(q)$,
- the embeddings of the edge group $G(p) \times G(q)$ into the vertex groups $G(p\lceil f \rceil q) = G(p) \times G(\lceil f \rceil) \times G(q)$ and $G(p\lfloor f \rfloor q) = G(p) \times G(\lfloor f \rfloor) \times G(q)$ are coordinate-wise,
- the group assigned to every 2-cell in $\mathrm{Sq}(\mathcal{K})$ is trivial.

Theorem 4 of [17] shows that this algebraic construction corresponds to the following topological construction provided all groups in $\mathcal{G}_X$ are diagram groups.

Let $\mathcal{K}$ be a directed 2-complex with the set of edges $\mathbf{E}$. For every $e \in \mathbf{E}$ let $\mathcal{K}_e$ be a directed 2-complex with a distinguished 1-path p. Attach each $\mathcal{K}_e$ to the edge e of $\mathcal{K}$ using the switch with edges e, b_e, c_e, s_e. As a result we obtain a new directed 2-complex $\overline{\mathcal{K}}$.

One can easily check that this definition coincides with the construction in [17, Theorem 4] if $\mathcal{K} = \mathcal{K}_{\mathcal{P}}$ for some $\mathcal{P}$. The only difference is that we used (in the new terminology) switches of the form $a = bsb$ instead of $a = bsc$. This difference is insignificant and the proof of [17, Theorem 4] can be easily generalized to arbitrary directed 2-complexes:

Theorem 9.1. *For every non-empty* 1*-path p in $\mathcal{K}$, the diagram group $\mathcal{D}(\overline{\mathcal{K}}, p)$ is isomorphic to the diagram product $\mathcal{D}(\mathcal{G}_{\mathbf{E}}; \mathcal{K}, p)$.*

For example (see [17]), if $\mathcal{K} = \langle e, e_i \ \ (i \in I) \mid e = e_i \ \ (i \in I) \rangle$, then the free product of the family of groups G_i $(i \in I)$ is the diagram product of the family $G_e = 1$, $G_{e_i} = G_i$ $(i \in I)$ over $\mathcal{K}$ with base e.

In order to show that the direct power of a diagram group is a diagram group ([17, Theorem 9]), take any diagram group G and let us consider the directed 2-complex $\mathcal{K} = \langle p, z \mid pz = p \rangle$ of Figure 17.

We proved in [17] that if $G_p = 1$, $G_z = G$, then the diagram product of this family of groups over $\mathcal{K}$ with base p is the infinite (countable) direct power of G.

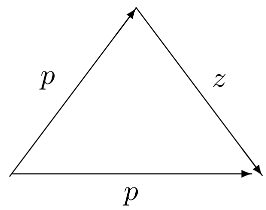

Figure 17.

Thus the class of diagram groups is closed under infinite (countable) direct powers. We have already noted that it is also closed under finite direct products. Now we obtain a stronger result.

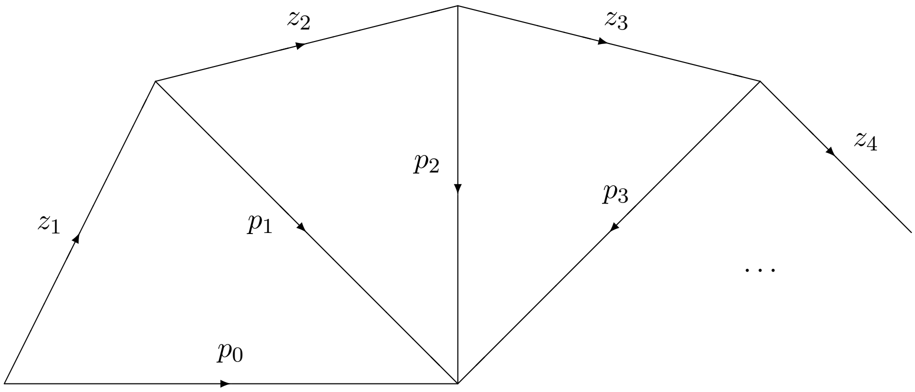

Figure 18.

Theorem 9.2. *The class of diagram groups is closed under any countable direct products.*

Proof. Let G_i $(i \geq 1)$ be a countable family of diagram groups. Let

$$\mathcal{K} = \langle p_0, z_1, p_1, z_2, p_2, \ldots \mid p_0 = z_1 p_1, p_1 = z_2 p_2, \ldots \rangle$$

(see Figure 18).

Now to any edge of $\mathcal{K}$ we assign a group. Let $G_{p_i} = 1$ for all $i \geq 0$ and let $G_{z_i} = G_i$ for all $i \geq 1$. This family of groups will be denoted by $\mathcal{G}$. We claim that the diagram product $\mathcal{D}(\mathcal{G}; \mathcal{K}, p_0)$ is isomorphic to the direct product $G_1 \times G_2 \times \cdots \times G_n \times \cdots$.

Indeed, the connected component $\mathcal{S} = \mathrm{Sq}(\mathcal{K}, p_0)$ consists of vertices (paths) that are homotopic to p_0. They are: $v_0 = p_0$, $v_1 = z_1 p_1$, $v_2 = z_1 z_2 p_2$, ..., $v_n = z_1 \cdots z_n p_n$, ... The edges are $e_n = (z_1 \cdots z_{n-1}, p_{n-1} = z_n p_n, 1)$ $(n \geq 1)$ and their inverses.

There are no 2-cells in this Squier complex because there are no independent pairs of edges. We choose an orientation on $\mathcal{S}$ making all the edges of the form e_n $(n \geq 1)$ positive. According to the definition, the diagram product $\mathcal{D}(\mathcal{G}; \mathcal{K}, p_0)$ is the fundamental group of the following graph of groups with the underlying graph $\mathcal{S}$ (see Figure 19).

To each vertex v of $\mathcal{S}$ we assign a group H_v, which is the direct product of the groups of $\mathcal{G}$ assigned to the letters of the word v. Obviously, $H_{v_0} = 1$, $H_{v_1} = G_1$, $H_{v_2} = G_1 \times G_2$, ..., $H_{v_n} = G_1 \times \cdots \times G_n$, ... Now we assign a group to each positive edge of $\mathcal{S}$. We have

$H_{e_i} = G_1 \times \cdots \times G_{i-1}$ for all $i \geq 1$. There are two natural embeddings of H_{e_i} into $H_{\iota(e_i)} = H_{v_{i-1}}$ and into $H_{\tau(e_i)} = H_{v_i}$. The first one is just identical, the second one embeds the direct product of the first $i-1$ groups of the family into the direct product of the first i groups ($i \geq 1$).

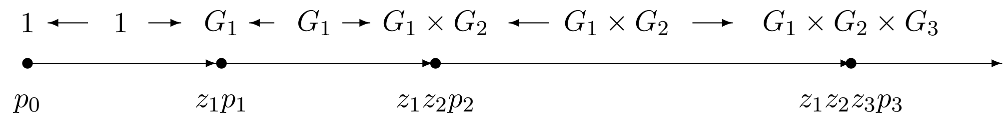

Figure 19.

The standard formulas for computing presentations of graphs of groups (see formulas (4), (5) from [17]) show that our diagram product is the free product of the groups H_{v_i} ($i \geq 0$) subject to the relations which simply mean that each of the groups from the following sequence of natural embeddings

$$1 \to G_1 \to G_1 \times G_2 \to \cdots \to G_1 \times \cdots \times G_n \to \cdots \to \mathcal{D}(\mathcal{G}; \mathcal{K}, p_0)$$

is identified with its image under the corresponding mapping, and $\mathcal{D}(\mathcal{G}; \mathcal{K}, p_0)$ is the inductive limit of this sequence of groups. Thus $\mathcal{D}(\mathcal{G}; \mathcal{K}, p_0)$ is isomorphic to the infinite direct product of groups $G_1 \times G_2 \times \cdots$, as desired. □

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