Relatively Hyperbolic Groups with Rapid Decay Property
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Abstract
We prove that a finitely generated group $G$ hyperbolic relative to the collection of finitely generated subgroups $\{H_1, \ldots, H_m\}$ has the Rapid Decay property if and only if each $H_i$, $i = 1, 2, \ldots, m$, has the Rapid Decay property.

1 Introduction
Throughout the whole paper, $G$ denotes a finitely generated group, $\{H_1, H_2, \ldots, H_m\}$ denotes a collection of finitely generated subgroups of $G$, and $1$ denotes the neutral element in $G$. We also denote $H_i \setminus \{1\}$ by $H_i^*$, for every $i = 1, 2, \ldots, m$.

The $L^2$-norm of a function $x$ in $l^2(G)$ is denoted by $\|x\|$.

A group $G$ satisfies the Rapid Decay property (RD property, in short) [J, C] if the space of rapidly decreasing functions on $G$ with respect to some length function is inside the reduced $C^*$-algebra of $G$ (see Section 2 for the precise definition). The RD property is relevant to the Novikov conjecture [CM] and to the Baum-Connes conjecture [L].

The main result of the paper is the following.

Theorem 1.1 (Theorem 3.1). Let $G$ be a group which is hyperbolic relative to the subgroups $\{H_1, H_2, \ldots, H_m\}$. Then $G$ has the RD property if and only if $\{H_1, \ldots, H_m\}$ have the RD property.

The “only if” part follows from the more general statement that a subgroup of a group that has the RD property also has the RD property with respect to the induced length-function [J, Proposition 2.1.1]. The “if” part is more difficult. A proof of it is given in Section 3.

Some particular cases of Theorem 1.1 have been proven before:

• All hyperbolic groups satisfy RD property [J, H, C] (this is a particular case of Theorem 1.1 with $m = 1$, $H_1 = \{1\}$).

• The amalgamated product of two groups $A$ and $B$ with finite amalgamated subgroup $F$ satisfies RD provided $A$ and $B$ satisfy RD [J] (take $G = A \star_F B$, $H_1 = A$, $H_2 = B$).

• A relatively hyperbolic group $G$ has RD property provided its parabolic subgroups $H_1, \ldots, H_m$ have polynomial growth [ChR]. The fact that polynomial growth implies RD follows from the definition of RD [J].

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2 Preliminaries

Recall that a length-function on a group $G$ is a map $L: G \to \mathbb{R}_+$ satisfying:

1. $L(gh) \leq L(g) + L(h)$ for all $g, h \in G$;
2. $L(g) = L(g^{-1})$ for all $g \in G$;
3. $L(1) = 0$.

Notations: We denote by $\mathcal{B}_L(r)$ the set $\{g \in G \mid L(g) \leq r\}$ and by $S_L(r)$ the set $\{g \in G \mid L(g) = r\}$.

We say that a length-function $L_1: G \to \mathbb{R}_+$ dominates another length-function $L_2: G \to \mathbb{R}_+$ if there exist $a, b \in \mathbb{R}_+$ such that $L_2 \leq aL_1 + b$. If $L_1$ dominates $L_2$ and $L_2$ dominates $L_1$ then $L_1$ and $L_2$ are said to be equivalent.

Remark 2.1. If $L_1$ dominates $L_2$ then there exists $c \in \mathbb{R}_+$ such that for every $r \geq 1$ we have $\mathcal{B}_{L_1}(r) \subset \mathcal{B}_{L_2}(cr)$.

If $G$ is a finitely generated group, the length-functions corresponding to two finite generating sets are equivalent. All such length-functions are called word length-functions.

Lemma 2.2 ([J], Lemma 1.1.4). If $G$ is a finitely generated group then any word length-function dominates all length-functions on $G$.

In order to recall the definition of the RD property for a length function $L$ we first need some notations. For every $s \in \mathbb{R}$ the Sobolev space of order $s$ with respect to $L$ is the set $H_L^s(G)$ of functions $x$ on $G$ such that $(1 + L)^s x$ is in $l^2(G)$. The space of rapidly decreasing functions on $G$ with respect to $L$ is the set $H_L^\infty(G) = \bigcap_{s \in \mathbb{R}} H_L^s(G)$. Thus in some sense a rapidly decreasing function decreases faster than any reciprocal of a polynomial in the length function.

The group algebra of $G$, denoted by $\mathbb{C}G$, is the set of functions with finite support on $G$, i.e. it is the set of formal linear combinations of elements of $G$ with complex coefficients. We denote by $\mathbb{R}_+G$ its subset consisting of functions taking values in $\mathbb{R}_+$.

With every element $g \in G$ we can associate the linear convolution operator $\phi \mapsto g \ast \phi$ on $l^2(G)$. Recall that

$$g \ast \phi(h) = \phi(g^{-1}h).$$

This induces an action of $G$ on $l^2(G)$ which can be extended to an action of $\mathbb{C}G$ on $l^2(G)$ by linearity. This action is faithful and every convolution operator induced by an element of $\mathbb{C}G$ is bounded. Therefore we can identify $\mathbb{C}G$ with a subspace in the space of bounded operators $\mathcal{B}(l^2(G))$ on $l^2(G)$.

For every $x \in \mathbb{C}G$ we denote by $\|x\|_*$ its operator norm, that is

$$\|x\|_* = \sup \{ \|x \ast \phi\| \mid \|\phi\| = 1 \} .$$

The closure $C_r^*(G)$ of $\mathbb{C}G$ in the operator norm is called the reduced $C^*$-algebra of $G$.

Definition 2.3. The group $G$ is said to have the RD property with respect to the length-function $L$ if $H_L^\infty(G) \subset C_r^*(G)$.

For details on this property we refer to [C], [J], [ChR].

We recall an equivalent way of defining the RD property. The following result is a slight modification of Proposition 1.4 in [ChR]. For this reason we provide a proof of it, although all the ideas are already in [ChR].
Lemma 2.4. Let $G$ be a discrete group endowed with a length-function $L$. The following are equivalent:

(1) The group $G$ has the RD property with respect to $L$.

(2) There exists a polynomial $P$ such that for every $r > 0$ and every $x \in \mathbb{R}_+G$ such that $x$ vanishes outside $\overline{B}_L(r)$, and every $\phi \in \ell^2(G)$ such that $\phi(G) \subseteq \mathbb{R}_+$, we have

$$ \|x \ast \phi\| \leq P(r) \|x\| \cdot \|\phi\|. \tag{1} $$

Proof. In [ChR, Proposition 1.4] it is proved that (1) is equivalent to (2) for $x \in \mathbb{R}_+G$ and $\phi \in \ell^2(G)$. We prove that if (2) is satisfied for $\phi$ with $\phi(G) \subseteq \mathbb{R}_+$ then it is satisfied for every $\phi \in \ell^2(G)$. Let $\phi \in \ell^2(G)$. Then we can write

$$ \phi = \phi_1 - \phi_2 + i(\phi_3 - \phi_4), $$

where $\phi_i$ take values in $\mathbb{R}_+$ and $\|\phi\|^2 = \sum_{i=1}^{4} \|\phi_i\|^2$. We have the inequalities

$$ \|x \ast \phi\| \leq \sum_{i=1}^{4} \|x \ast \phi_i\| \leq P(r) \|x\| \sum_{i=1}^{4} \|\phi_i\| = P(r) \|x\| \cdot \|\phi\|. $$

\hfill \square

Definition 2.5. A group is said to have the RD property if there is a length-function on it such that the group satisfies the RD property with respect to it.

Remark 2.6. Lemma 2.2 and Remark 2.1 imply that if a finitely generated group satisfies RD with respect to some length-function, then it satisfies RD with respect to a word length-function (for some generating set). Hence, a group satisfies RD if and only if it satisfies RD with respect to every word length-function.

We recall some useful properties of relatively hyperbolic groups proved in [DS]. Let $G$ be a group which is hyperbolic relative to the subgroups $\{H_1, H_2, \ldots, H_m\}$.

Notations: For a subset $A$ in a metric space we denote by $N_\delta(A)$ the tubular neighborhood of $A$, that is $\{x \mid \text{dist}(x, A) < \delta\}$, and by $\overline{N_\delta(A)}$ its closure, that is $\{x \mid \text{dist}(x, A) \leq \delta\}$.

Proposition 2.7 ([DS], Corollary 8.14, Lemma 8.19). There exists $\kappa > 0$ such that the following holds:

(1) For every triangle with geodesic edges in the Cayley graph of $G$, one of the following two situations occurs.

(C) There exists $g \in G$ such that $B(g, \kappa)$ intersects each of the three edges;

(P) There exists a unique left coset $gH_i$, $g \in G$, $i \in \{1, 2, \ldots, m\}$, such that $N_\kappa(gH_i)$ intersects each of the three edges.

(2) Let $g_1, g_2$ be two geodesics in the Cayley graph of $G$ joining a point $a$ with $b_1, b_2$, respectively, such that $b_i \in \overline{N_\kappa(gH)}$ for some $g \in G$ and $H \in \{H_1, \ldots, H_m\}$ and $g_i \cap \overline{N_\kappa(gH)} = \{b_i\}$ for $i = 1, 2$. Then $\text{dist}(b_1, b_2) \leq D$, where $D = D(\kappa)$.

\footnote{We only consider the strong relative hyperbolicity in this paper. For a definition of strong relative hyperbolicity see [DS] and references therein. In fact we only use properties of relative hyperbolicity listed in Proposition 2.7.}
3 Proof of the main result

Theorem 3.1. Let $G$ be a group which is hyperbolic relative to the subgroups $\{H_1, H_2, \ldots, H_m\}$. Then $G$ has the RD property if and only if $\{H_1, \ldots, H_m\}$ have the RD property.

Proof. We prove the “if” part. Consider a word length-function $L$ on $G$. According to the argument in the proof of Theorem 5, [C, §III.5.a], it suffices to show the inequality (1) for $x \in \mathbb{R}_+G$ which vanishes outside $S_L(r)$, $y \in \mathbb{R}_+G$ which vanishes outside $S_L(R)$ and for $(x * \phi)_g$ instead of $x * \phi$, where $r$ and $R$ are arbitrary positive real numbers, while $g \in [R - r, R + r]$, and where for every $f \in l^2(G)$, $f_\phi$ denotes the function which coincides with $f$ on $S_L(g)$ and which vanishes outside $S_L(g)$. In what follows we consider the three positive constants $r$, $R$ and $\rho$ fixed.

Let $P_i(r)$ be the polynomial given by Lemma 2.4, (2), for the group $H_i$, $i = 1, 2, \ldots, m$, and let $P(r) = 1 + \sum_{i=1}^m P_i(r)^2$. We shall prove that for every $g \in [R - r, R + r]$ we have that

$$
\|(x * \phi)_g\|^2 \leq Q(r)P(r)\|x\|^2 \cdot \|\phi\|^2,
$$

(2)

where $Q$ is a polynomial of degree 3.

For every $g \in S_L(\lambda)$ we choose one geodesic $q_g$ joining it to 1. We thus obtain a set of geodesics of length $\lambda$ indexed by elements in $S_L(\lambda)$. We denote this set of geodesics $G(\lambda)$ and we identify it with the set $S_L(\lambda)$.

Consider a geodesic $q_g$ in $G(g)$, and an arbitrary geodesic triangle with $q_g$ as an edge and the other two edges $q_h \in G(r)$ and $q_k \in G(R)$ such a triangle corresponds to a decomposition $g = hk$, where $h \in S_L(r)$ and $k \in S_L(R)$. According to Proposition 2.7, (1), the geodesic triangle satisfies conditions of either case (C) or case (P).

In case (C) there exists $g_1$ such that $B(g_1, \kappa)$ intersects the three edges. Consequently we can write $g = g_1g_2$, $h = g_1g_3$ and $k = g_3^{-1}g_2$, and $g_1$ is at distance at most $\kappa$ from $q_g, q_h, q_k$.

In case (P) there exists $\gamma H_i$, $i \in \{1, 2, \ldots, m\}$, such that $N_\kappa(\gamma H_i)$ intersects the three edges. Hence we can write $g = g_1\eta g_2$, where $g_1 \in \gamma H_i$, $g_1$ at distance at most $\kappa$ from the entrance point of $q_g$ in $N_\kappa(\gamma H_i)$, $\eta \in H_i^\ast$ and $g_1\eta$ is at distance at most $\kappa$ from the exit point of $q_g$ from $N_\kappa(\gamma H_i)$. Proposition 2.7, (2), implies that $g_1$ is at distance at most $D + \kappa$ of the entrance point of $q_g$ in $N_\kappa(\gamma H_i)$ and that $g_1\eta$ is at distance at most $D + \kappa$ of the exit point of $q_k$ from $N_\kappa(\gamma H_i)$. It follows that there exists $\eta' \in H_i^\ast$ such that $g_1\eta'$ is at distance at most $\kappa$ from the exit point
of $q_h$ from $\mathcal{N}_\kappa(\gamma H_i)$ and at distance at most $D + \kappa$ of the entrance point of $q_k$ in $\mathcal{N}_\kappa(\gamma H_i)$. If we denote by $\eta''$ the element $(\eta')^{-1}\eta$ then we have that $h = q_1^\eta g_3$ and that $k = q_2^{-1}\eta'' g_2$. We also note that if one of the entrance/exit points of one geodesic in/from $\mathcal{N}_\kappa(\gamma H_i)$ does not exist, then it is replaced by the corresponding endpoint of the geodesic.

A similar argument can be written for an arbitrary geodesic $q_g$ in $G(R)$, and a geodesic triangle with $q_g$ as an edge and the other two edges $q_h \in G(r)$ and $q_k \in G(g)$.

Notation: We denote by $\sigma$ the constant $\kappa + D(\kappa)$, with $\kappa$ given by Proposition 2.7, (1), and $D(\kappa)$ given by Proposition 2.7, (2).

The previous considerations justify the following notations and definitions.

Notations:

(a) We denote by $\Delta_\eta$ the set composed of all $(g_1, g_2, g_3, \eta, \eta', \eta'') \in G^3 \times \bigcup_{i=1}^m (H_i^*)^3$ such that:

1. $g = g_1 \eta g_2 \in S_L(g)$, $h = g_1 \eta' g_3 \in S_L(r)$ and $k = g_3^{-1} \eta'' g_2 \in S_L(R)$;
2. $\eta = \eta' \eta''$;
3. Suppose that $(\eta, \eta', \eta'') \in (H_i^*)^3$, $i \in \{1, 2, \ldots, m\}$. Then the following holds:
   * $g_1$ is at distance at most $\sigma$ from the entrance points of $q_g$ and $q_h$ in $\mathcal{N}_\kappa(g_1 H_i)$;
   * $g_1 \eta$ is at distance at most $\sigma$ from the exit points of $q_g$ and $q_k$ from $\mathcal{N}_\kappa(g_1 H_i)$;
   * $g_1 \eta'$ is at distance at most $\sigma$ from the exit point of $q_h$ from $\mathcal{N}_\kappa(g_1 H_i)$ and from the entrance point of $q_k$ in $\mathcal{N}_\kappa(g_1 H_i)$.

(b) We denote by $\Delta_\kappa$ the set consisting of all $(g_1, g_2, g_3, 1, 1, 1) \in G^3 \times \{1\}^3$ such that:

1. $g = g_1 g_2 \in S_L(g)$, $h = g_1 g_3 \in S_L(r)$ and $k = g_3^{-1} g_2 \in S_L(R)$;
2. $g_1$ is at distance at most $\kappa$ from $q_g, q_h$ and $q_k$.

Definition 3.2. The set $\Delta = \Delta_\eta \cup \Delta_\kappa$ is called the set of central decompositions of geodesic triangles with edges in $G(g) \times G(r) \times G(R)$.

Definitions 3.3. Given a geodesic $q_g$ in $G(\lambda)$ with $\lambda \in \{g, R\}$, we call every decomposition $g = g_1 g_2$ or $g = g_1 \eta g_2$ of $g$ corresponding to a central decomposition in $\Delta$ a central decomposition of $g$. We call $g_1$ and $g_2$ the left and right parts of the decomposition. Either $\eta$ in the second case or $1$ in the first case is called the middle part of the decomposition of $g$.

For the fixed $g$ (and $q_g$) we denote by $L_g$ the set of left parts of central decompositions of $g$, by $R_g$ the set of right parts of central decompositions of $g$ and by $D_g$ the set of triples $(g_1, \eta, g_2)$ corresponding to central decompositions of $g$. We also denote by $L_g R_g$ the set of pairs of left and right parts that can appear in a central decomposition of $g$.

We denote by $D_g$ the set of all $D_g$ with $g \in S_L(g)$. The sets $L_g R_g, L_g, R_g$ are defined similarly.

Notations: Let $C$ be a subset of $\prod_{i=1}^n X_i$. Let $a_i$ be a point in the $i$-th projection of $C$, let $I$ be a subset in $\{1, 2, \ldots, n\} \setminus \{i\}$ and let $X^I = \prod_{i \in I} X_i$. We denote by $C^I(a_i)$ the set of elements $\bar{x}$ in $X^I$ such that $(\bar{x}, a_i)$ appears in the projection of $C$ in $X^I \times X_i$. Whenever there is no risk of confusion, we drop the index $I$ in $C^I(a_i)$.

For every fixed decomposition $d = (g_1, \eta, g_2) \in D_g$ we also denote by $C_d$ the set $\Delta(d)$, that is, the set of triples $(\eta', \eta'', g_3)$ such that $(g_1, g_2, g_3, \eta, \eta', \eta'')$ is in $\Delta$. We denote by $U_d$ the set of
Every \( g \in S_L(\lambda) \), with \( \lambda \in \{ g, R \} \), has at most \( C_1 r + C_2 \) central decompositions, where \( C_1 \) and \( C_2 \) are universal constants.

**Proof.** Suppose that \( g_1 \) is the left part of a central decomposition of \( g \) with nontrivial middle part. Then \( g_1 \) is at distance at most \( \sigma \) from a point in a geodesic of length \( r \). It follows that \( g_1 \) has length at most \( r + \sigma \). On the other hand \( g_1 \) is at distance at most \( \sigma \) from a vertex \( q_1 \) of \( g \), whence \( L(g_1) \leq r + 2\sigma \). Thus, we have at most \( r + 2\sigma \) possibilities for \( g_1 \). For each such \( g_1 \) there are finitely many left cosets \( \gamma H_i \) at distance at most \( \kappa \) from it, so \( g_1 \) is the entrance point of \( q_g \) in \( \mathcal{N}_\kappa(\gamma H_i) \) for finitely many left cosets \( \gamma H_i \). The exit point \( g_2 \) of \( q_g \) from \( \mathcal{N}_\kappa(\gamma H_i) \) is uniquely defined each time the left coset is fixed. For each left coset, there are finitely many points in it at distance at most \( \sigma \) from \( g_1 \), and likewise for \( g_2 \). Thus, there are finitely many possibilities for \( g_1 \) and \( g_1 q \), once \( g_1 \) is fixed. We deduce that on the whole there are at most \( K(r + 2\sigma) \) possible central decompositions of \( g \) with middle part non-trivial.

A similar but much easier argument allows to deduce the same conclusion for central decompositions with trivial middle part.

\[ \Box \]

We continue the proof of Theorem 3.1. We can write that
\[
\| (x * \phi)_\bar{g} \|^2 = \sum_{g \in S_L(\bar{g})} [(x * \phi)(g)]^2 = \sum_{g \in S_L(\bar{g})} \left[ \sum_{(h,k) \in S_L(r) \times S_L(R), hk = g} x(h)\phi(k) \right]^2 \leq \sum_{g \in S_L(\bar{g})} \left[ \sum_{d \in \mathcal{D}_g (\eta_1, \eta_2, g_3) \in C_d} \sum_{\eta \in S_L(\bar{g})} x(g_1 \eta g_3)\phi(g_3^{-1} \eta'' g_2) \right]^2.
\]

This inequality applied in the last term of (3) to the first sum in the brackets, together with Lemma 3.4, gives
\[
\| (x * \phi)_\bar{g} \|^2 \leq (C_1 r + C_2) \sum_{g \in S_L(\bar{g})} \sum_{d \in \mathcal{D}_g} \left[ \sum_{(\eta_1, \eta_2, g_3) \in C_d} x(g_1 \eta g_3)\phi(g_3^{-1} \eta'' g_2) \right]^2.
\]

We re-write the term in the brackets on the right side of inequality (5) and apply the Cauchy-Schwarz inequality as follows:
\[
\sum_{(\eta_1, \eta_2, g_3) \in C_d} x(g_1 \eta g_3)\phi(g_3^{-1} \eta'' g_2) = \sum_{e = (\eta', \eta'')} \sum_{g_3 \in C_d(e)} x(g_1 \eta g_3)\phi(g_3^{-1} \eta'' g_2) \leq \sum_{e = (\eta', \eta'')} \left[ \sum_{g_3 \in C_d(e)} (x(g_1 \eta g_3))^2 \right]^{1/2} \left[ \sum_{g_3 \in C_d(e)} (\phi(g_3^{-1} \eta'' g_2))^2 \right]^{1/2}.
\]

We now define, for every \( g \in S_L(\bar{g}) \), \( \bar{g} \in \mathcal{L} R_g \) and \( i \in \{1, 2, \ldots, m\} \) the function \( X^i_{\bar{g}} : H_i \rightarrow \mathbb{R}_+ \) by
\[
X^i_{\bar{g}}(\eta') = \left[ \sum_{g_3 \in C_d(\eta', \eta'' g_2)} (x(g_1 \eta' g_3))^2 \right]^{1/2} \text{ for every } \eta \in \mathcal{D}_g(\bar{g}) \cap H_i^* \text{ and } \eta' \in E_{(\bar{g}, \eta)},
\]
and $X^i_g(\eta') = 0$ for all the other $\eta' \in H_i$.

Likewise we define the function $Y^i_g : H_i \to \mathbb{R}_+$ by

$$Y^i_g(\eta'') = \left[ \sum_{g \in C(g,n)} (\phi(g^{-1}_g g_3))^2 \right]^{1/2} \quad \text{for every } \eta \in D_g \cap H^*_i \text{ and } \eta'' \in E'_{g,n},$$

and $Y^i_g(\eta'') = 0$ for all the other $\eta'' \in H_i$.

Then the sum in (6), if the middle part $\eta$ of $d$ is in $H^*_i$, can be written as

$$\sum_{(\eta',\eta'') \in D_d \cap H^*_i \times H^*_i} X^i_g(\eta') Y^i_g(\eta'').$$

Thus, inequality (5) gives

$$\| (x \ast \phi)_{g} \|^2 \leq (C_1 r + C_2) \sum_{g \in S_L(\rho)} \sum_{\bar{g} \in L_{R_{g}}} \sum_{i=1}^{m} \| X^i_g \|^2 \| Y^i_g \|^2 \leq$$

$$P(r) \sum_{g \in S_L(\rho)} \sum_{\bar{g} \in L_{R_{g}}} \sum_{i=1}^{m} \| X^i_g \|^2 \| Y^i_g \|^2.\tag{7}$$

We denote by $S_1$ and $S_2$ the first and respectively the second sum in the second term of the inequality (7), without the factor $C_1 r + C_2$. We have

$$S_1 \leq \sum_{g \in S_L(\rho)} \sum_{\bar{g} \in L_{R_{g}}} \sum_{i=1}^{m} \| X^i_g \|^2 \leq \sum_{g \in S_L(\rho)} \sum_{\bar{g} \in L_{R_{g}}} \sum_{i=1}^{m} P(r)^2 \| X^i_g \|^2 \| Y^i_g \|^2 \leq$$

$$P(r) \sum_{g \in S_L(\rho)} \sum_{\bar{g} \in L_{R_{g}}} \sum_{i=1}^{m} \| X^i_g \|^2 \| Y^i_g \|^2.\tag{7}$$

The latter sum can be written as

$$\sum_{\bar{g} \in L_{R_{g}}} \left[ \sum_{\eta \in D_{g} \cap H^*_i} \sum_{g_{3} \in C(g,n)} \left( x(g_{1} g_{3} g_{2}) \right)^2 \right] \cdot \left[ \sum_{\eta \in D_{g} \cap H^*_i} \sum_{g_{3} \in C(g,n)} \left( x(g_{1} g_{3} g_{2}) \right)^2 \right] \cdot \left[ \sum_{g_{2} \in R_{g}} \sum_{g_{3} \in \Delta(g_{2})} \left( \phi(g_{2}^{-1} g_{3}) \right)^2 \right].$$

Every $x(\eta^2)$ with $g \in S_L(\rho)$ appears at most $K_1 r$ times in the first sum above, where $K_1$ is an universal constant, hence the first sum is at most $K_1 r \| x \|^2$. Lemma 3.4 applied to every $g \in S_L(\rho)$ implies that every $\phi(g)^2$ with $g \in S_L(\rho)$ appears at most $C_1 r + C_2$ times in the second sum above, hence the second sum is at most $(C_1 r + C_2) \| \phi \|^2$.

We deduce that

$$S_1 \leq Q_1(r) P(r) \| x \|^2 \| \phi \|^2,$$

where $Q_1$ is an universal polynomial of degree 2.

The sum $S_2$ can be written as

$$\sum_{\bar{g} \in L_{R_{g}}} \sum_{g_{3} \in \Delta(g_{2},1,1,1)} \left( x(g_{1} g_{3} g_{2}) \right)^2 \cdot \sum_{g_{2} \in R_{g}} \sum_{g_{3} \in \Delta(g_{2},1,1,1)} \left( \phi(g_{2}^{-1} g_{3}) \right)^2 \leq$$

$$\sum_{g_{1} \in L_{g}} \sum_{g_{2} \in \Delta(g_{2},1,1,1)} \left( x(g_{1} g_{3} g_{2}) \right)^2 \cdot \sum_{g_{2} \in R_{g}} \sum_{g_{3} \in \Delta(g_{2},1,1,1)} \left( \phi(g_{2}^{-1} g_{3}) \right)^2.$$

An argument as above now gives that

$$S_2 \leq Q_2(r) \| x \|^2 \| \phi \|^2,$$
where $Q_2$ is an universal polynomial of degree 2.

We may conclude that

$$\|(x \ast \phi)_{\varphi}\|^2 \le Q(r)P(r)\|x\|^2\|\phi\|^2,$$

where $Q$ is an universal polynomial of degree 3.

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