Groups acting on tree-graded spaces and splittings of relatively hyperbolic groups

Cornelia Druțu and Mark Sapir

$\mathbb{R}$-trees

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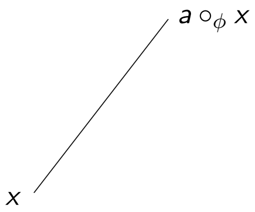

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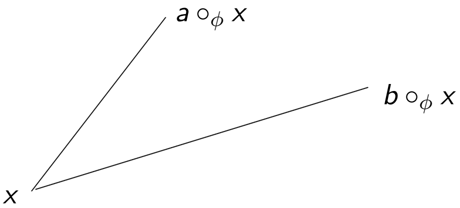

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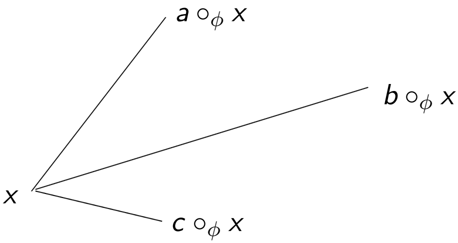

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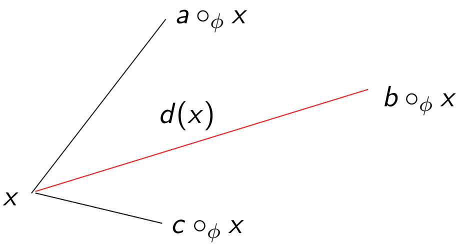

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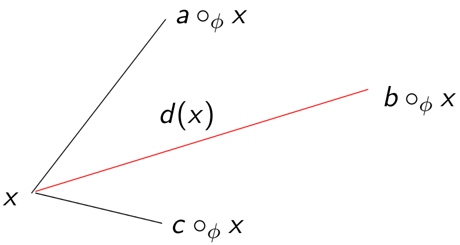

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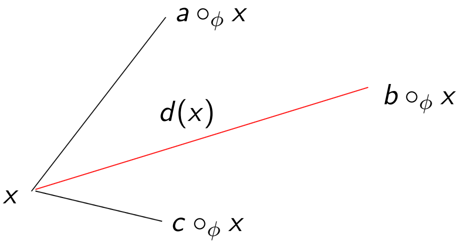

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The word many means that the translation numbers d_ϕ are unbounded.
Then we can divide the metric in X by d_ϕ, obtaining X_ϕ, $\phi\colon \Lambda \to G$. The $\mathbb{R}$-tree is the limit $\mathrm{Con}(X, (d_\phi), (x_\phi))$.

Applications

Given an action on an $\mathbb{R}$-tree, we can apply Rips -Bestvina - Feighn - Levitt -Sela - Guirardel... and split the group into a graph of groups.

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The asymptotic cones of non-hyperbolic spaces need not be trees.

But in many cases they are tree-graded spaces. Recall the definition (this is the same as *spaces of type* T_2 *in Kapovich-Kleiner-Leeb*).

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Then we say that the space $\mathbb{F}$ is *tree-graded with respect to* $\mathcal{P}$.

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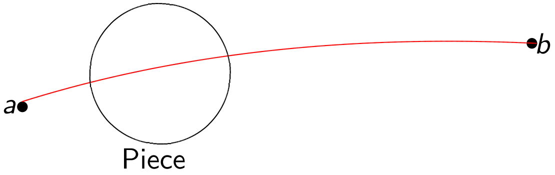

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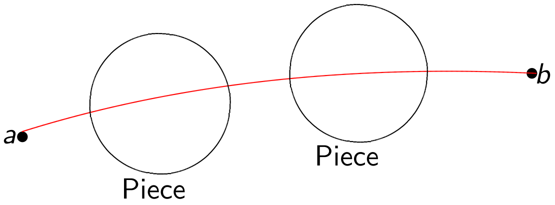

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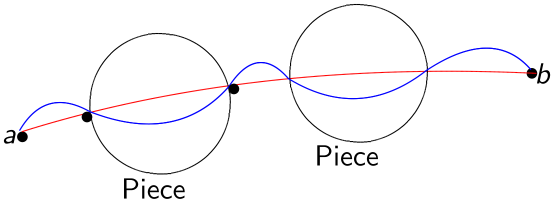

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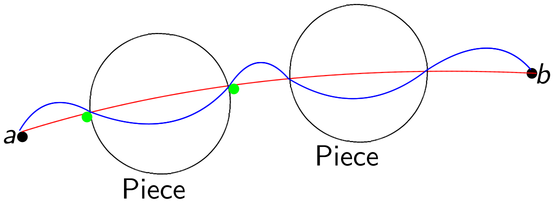

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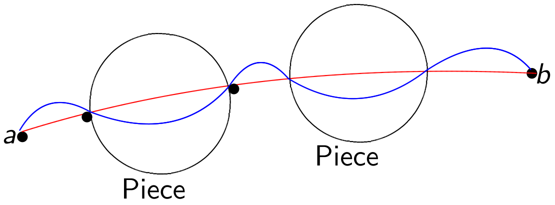

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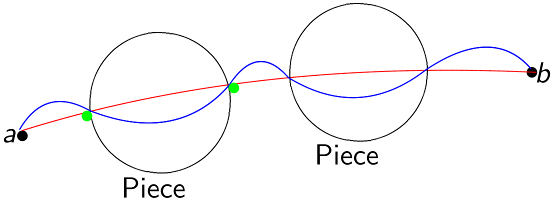

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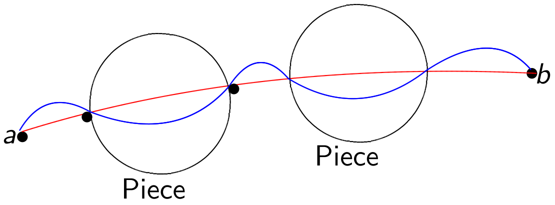

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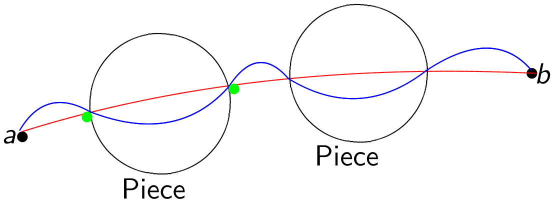

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Cut points and tree-graded structures

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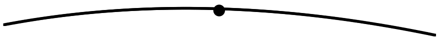

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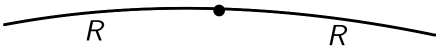

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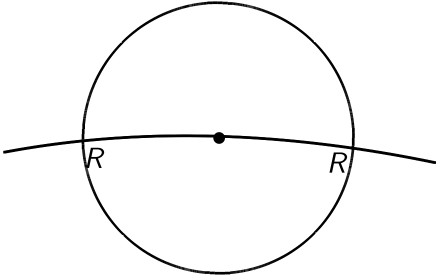

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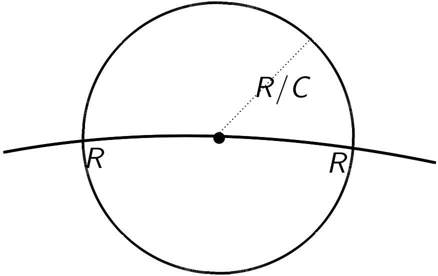

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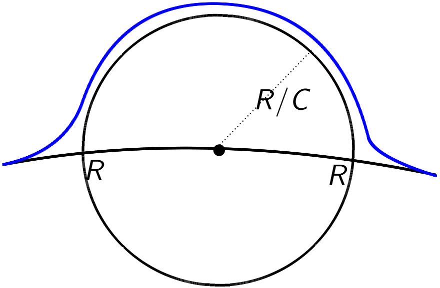

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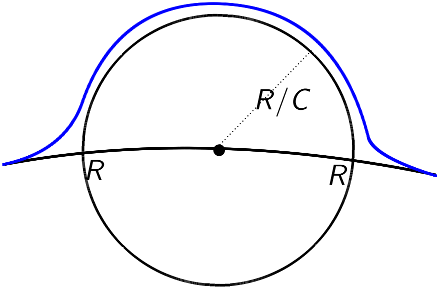

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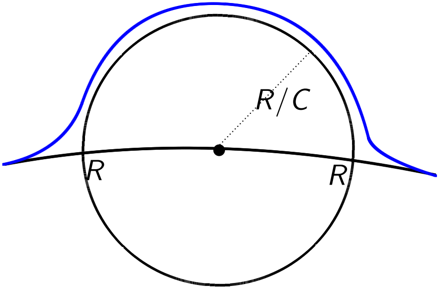

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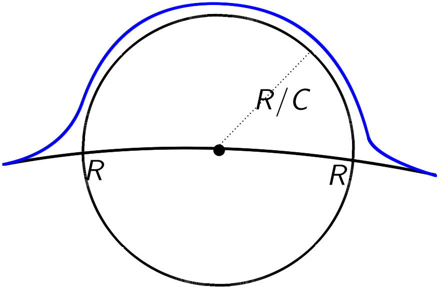

The length of the blue arc should be $> O(R)$.

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Recall that hyperbolicity $\equiv$superlinear divergence of any pair of geodesic rays with common origin.

Transversal trees

Definition. For every point x in a tree-graded space $(\mathbb{F}, \mathcal{P})$, the union of geodesics $[x, y]$ intersecting every piece by at most one point is an $\mathbb{R}$-tree called a *transversal* tree of $\mathbb{F}$.

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The geodesics $[x, y]$ from transversal trees are called *transversal geodesics*.

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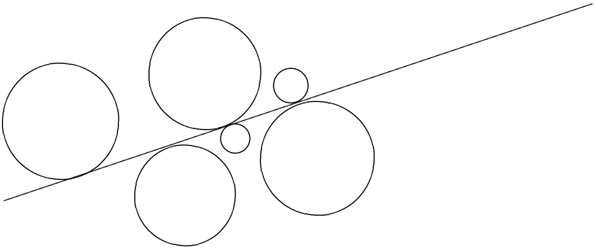

A tree-graded space. Pieces are the circles and the points on the line.

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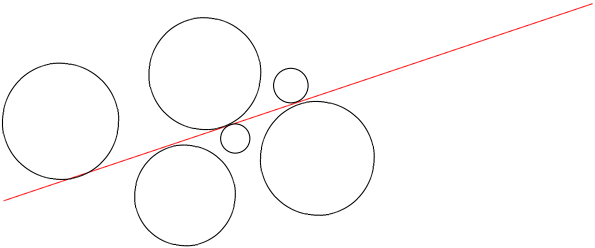

A tree-graded space. Pieces are the circles and the points on the line.

The line is a transversal tree, the other transversal trees are points on the circles.

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Statement 2. Let $\mathbb{F} = (X_n, \mathcal{P}_n)$ be a sequence of tree-graded spaces, ω be a ultrafilter.Let $\lim^{\omega}(X_n, o_n)$ be the ω-limit of X_n with observation points o_n.

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Statement 2. Let $\mathbb{F} = (X_n, \mathcal{P}_n)$ be a sequence of tree-graded spaces, ω be a ultrafilter.Let $\lim^\omega (X_n, o_n)$ be the ω-limit of X_n with observation points o_n. Let $\tilde{\mathcal{P}}$ be the set of ω-limits $\lim^\omega (M_n)$ where $M_n \in \mathcal{P}_n$. Then $\tilde{\mathcal{P}}$ is tree-graded with respect to $\tilde{\mathcal{P}}$.

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- (Olshanskii - S.) There exists a f.g. group such that one asymptotic cone has cut points and another one does not.
- Question (S): is there an amenable (non-virtually cyclic) group with cut points in an asymptotic cones?

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Our main result shows that a group acting "nicely" on a tree-graded space also acts "nicely" on an $\mathbb{R}$-tree.

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Theorem Let G be a finitely presented group acting on a tree-graded space $(\mathbb{F}, \mathcal{P})$. Suppose that the following hold:

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(3) T is a line and G has a subgroup of index at most 2 that is an extension of the kernel of that action by a finitely generated free Abelian group.

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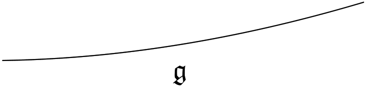
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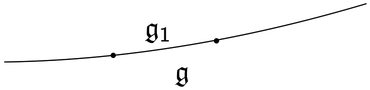
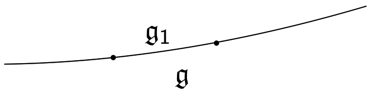

$\mathfrak{g}_1$
$\mathfrak{g}$



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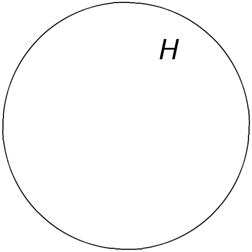

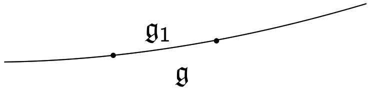

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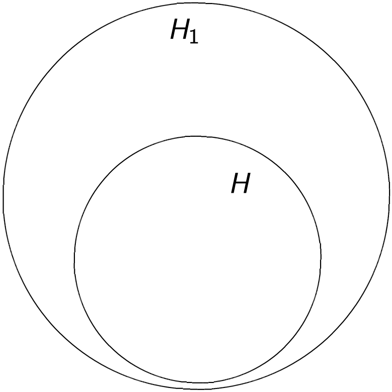

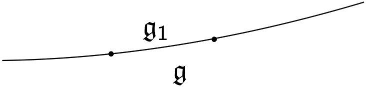

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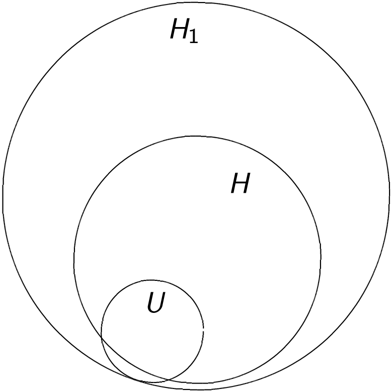

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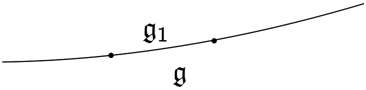

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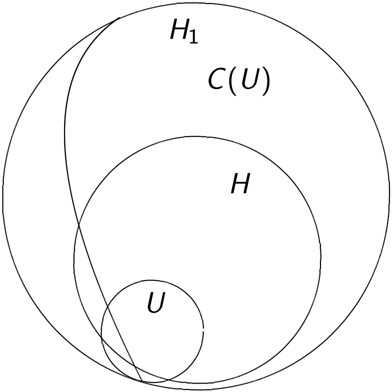

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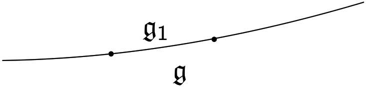

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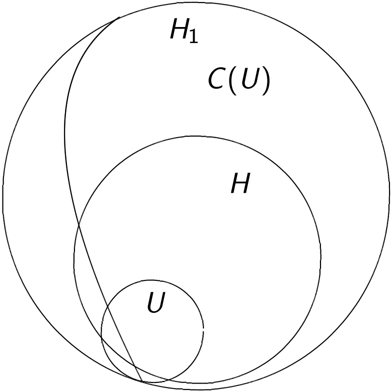

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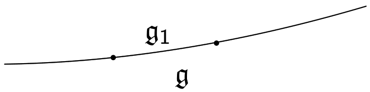

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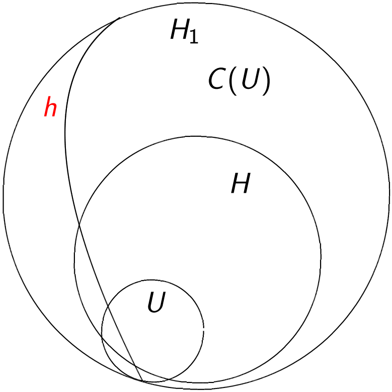

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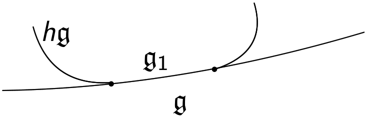

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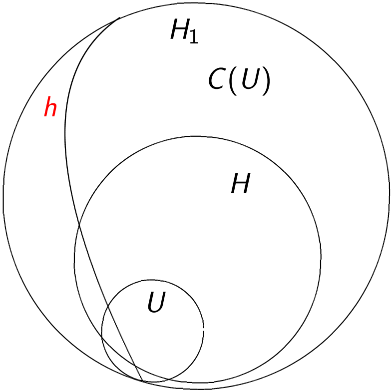

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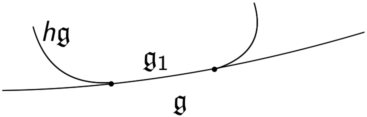

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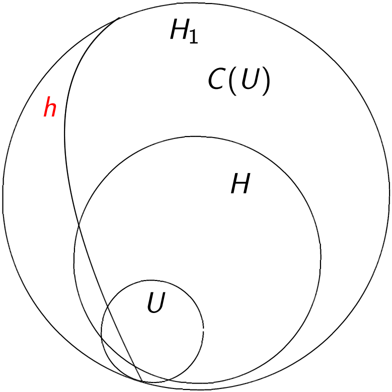

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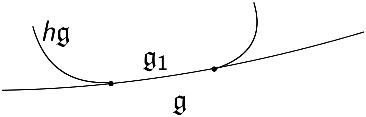

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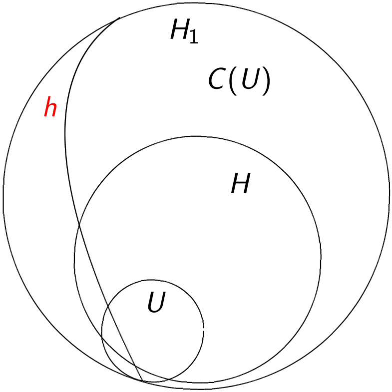

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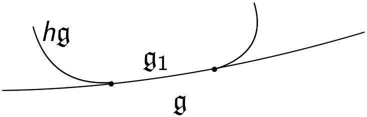

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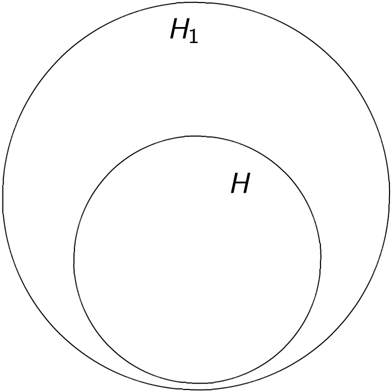

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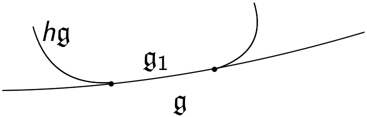

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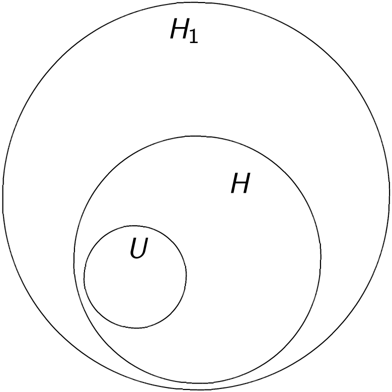

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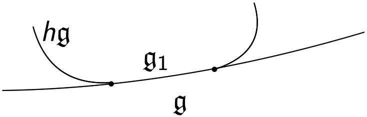

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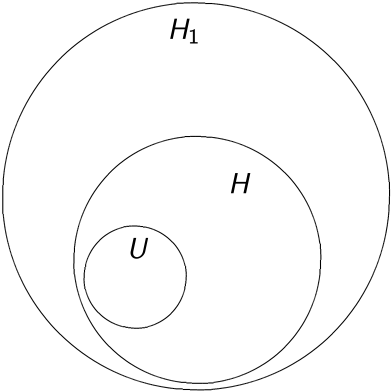

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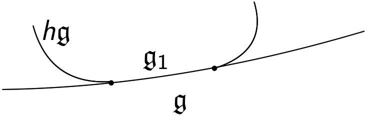

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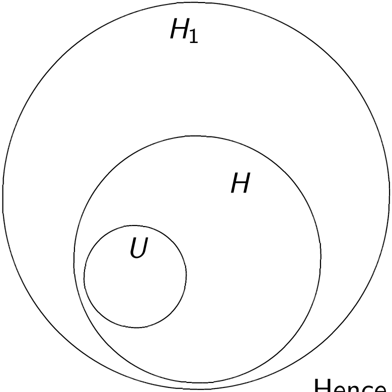

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Hence $D > |H \cap hHh^{-1}| = |H| > (D+1)!$.

Dahmani's result

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Theorem (Dahmani) If Λ is finitely presented, and G is relatively hyperbolic then there are finitely many subgroups of G, up to conjugacy, that are images of Λ in G by homomorphisms without accidental parabolics.

Homomorphisms into groups

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Note that if a group G splits over an Abelian subgroup C, say, $G = A *_C B$, then it typically has many outer automorphisms that are identity on A and conjugate B by elements of C. Hence we need to modify the definition of accidental parabolics as follows.

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Theorem Let Λ be a finitely generated group, G be a relatively hyperbolic group and parabolic subgroups are small (no free non-Abelian subgroups).

Then the number of pairwise non-conjugate in G injective homomorphisms $\Lambda \to G$ without weakly accidental parabolics is finite.

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- G splits over a virtually cyclic subgroup;
- G splits over a parabolic (finite of uniformly bounded size)-by-Abelian-by-(virtually cyclic) subgroup;
- G can be represented as a non-trivial amalgamated product or HNN extension with one of the vertex groups a maximal parabolic subgroup of G.

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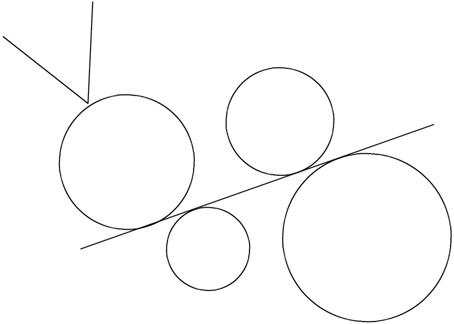

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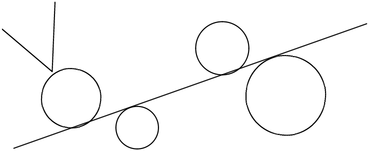

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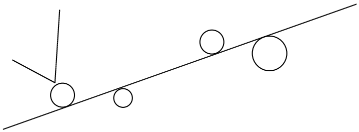

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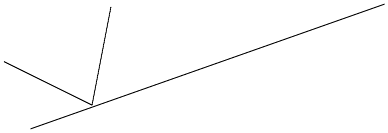

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Note that pieces do not intersect.

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Thus in this case G acts non-trivially on an $\mathbb{R}$-tree with arc stabilizers from $\mathcal{C}_2$.

Case B.

Suppose that G fixes a point in T.

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The corresponding $\approx$-class is a union of pieces and is a tree-graded space $(R, \mathcal{R})$ with trivial transversal trees. G acts on R.

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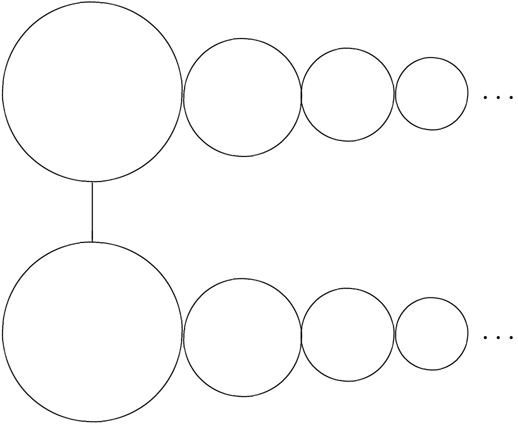

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Then we define a simplicial tree having pieces of $\mathcal{P}_{\delta-1}$ and intersections of these pieces as vertices, and edges connecting a piece and a vertex inside it.

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In case $\delta > 1$, the edge stabilizers are in $\mathcal{C}_1$.

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G does not fix a point in $\mathcal{P}_\alpha$.

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We define the structure of a pre-tree (Bowditch) on X.

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- (PT3) xzy and $z \neq w$ then $(xzw \vee yzw)$.

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We apply a version of Levitt's theorem and complete the proof.