GEOMETRY OF GROUPS AND COMPUTATIONAL COMPLEXITY

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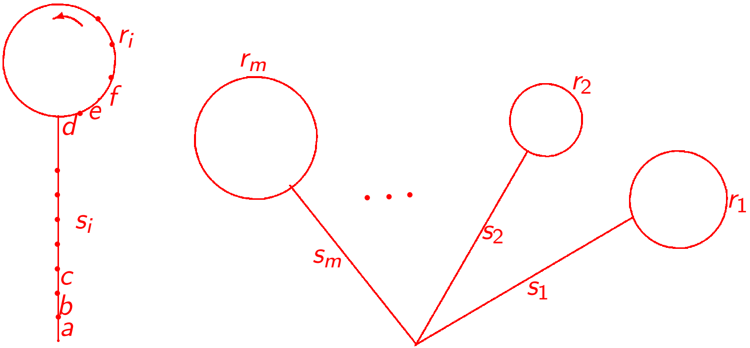

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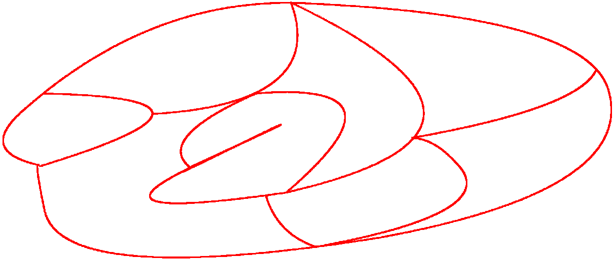

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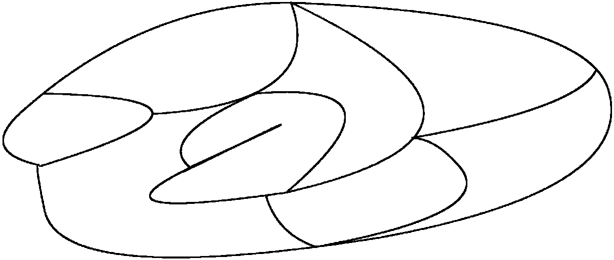

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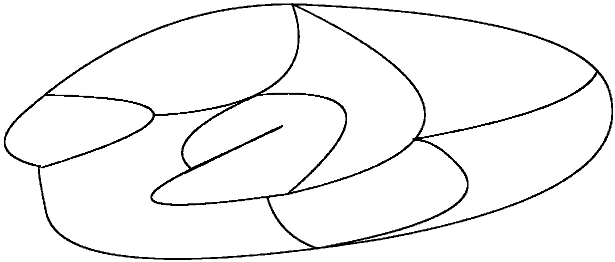

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The direct part of van Kampen lemma: If $w = 1$ in G, then there is a van Kampen diagram Δ over the presentation of G with boundary label w.

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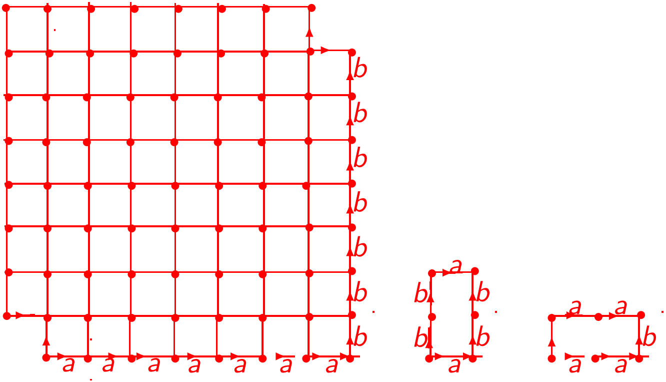

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The converse part of the van Kampen lemma

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1. $r_i \in R$; $u_1 u_2 ... u_{m+1} = 1$ in the free group;
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In particular the word problem is decidable if and only if the Dehn function is bounded by a recursive function.

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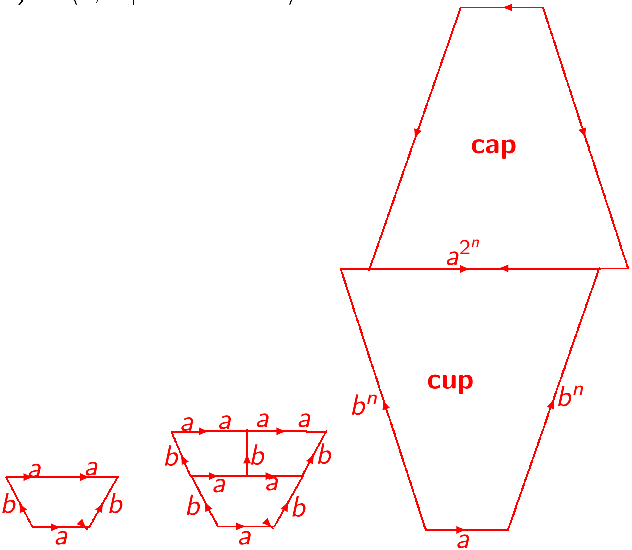

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Theorem (Birget, Olshanskii, Rips, S., Ann. of Math. 2002)
The word problem of a finitely generated group G is in NP if and only if G is embedded into a finitely presented group with polynomial Dehn function.

A connection with the classical isoperimetric problem for manifolds

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Theorem (R. Young, solving a conjecture of Thurston): The group $SL(n, \mathbb{Z})$ has quadratic Dehn function if $n \geq 5$.

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Theorem (S. Wenger): There are nilpotent groups with Dehn functions not of the form n^k for any k (bigger than n^2 and smaller than $n^2 \log n$.

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Question: Is it true that every NP-group is inside a group with simply connected asymptotic cones?

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Examples: $SL_n(\mathbb{Z})$, $n \geq 5$, the CAT(0)-groups, automatic groups, the R. Thompson group, etc.

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Examples. $\pi + e$, any algebraic number > 4, etc.

Some weird Dehn functions

Theorem (Olshanskii, S.) There exists a finitely presented group with non-recursive word problem and almost quadratic Dehn function.

Thank you!