Groups acting on tree-graded spaces and splittings of relatively hyperbolic groups

Cornelia Druțu and Mark Sapir
Observation due to Bestvina and Paulin: if a group has many actions on a Gromov-hyperbolic metric space then it acts non-trivially (i.e. without a global fixed point) by isometries on the asymptotic cone of that space which is an $\mathbb{R}$-tree.
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Then we can divide the metric in $X$ by $d_\phi$, obtaining $X_\phi$, $\phi: \Lambda \to G$. The $\mathbb{R}$-tree is the limit $\text{Con}(X,(d_\phi),(x_\phi))$. 

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The asymptotic cones of non-hyperbolic spaces need not be trees.

But in many cases they are tree-graded spaces. Recall the definition.
Tree-graded spaces

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Then we say that the space $\mathbb{F}$ is *tree-graded with respect to* $\mathcal{P}$. 
The main property of tree-graded spaces

For every:

- two points $a, b$ in $X$, 

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Cut points and tree-graded structures

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![Diagram](image-url)
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The length of the blue arc should be $> O(R)$. 
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Recall that hyperbolicity \equiv
Recall that hyperbolicity ≡ superlinear divergence of any pair of geodesic rays with common origin.
Transversal trees

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The geodesics \([x, y]\) from transversal trees are called *transversal geodesics*. 
Transversal trees, an example
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A tree-graded space. Pieces are the circles and the points on the line.
Transversal trees, an example

A tree-graded space. Pieces are the circles and the points on the line.
The line is a transversal tree, the other transversal trees are points on the circles.
**Cut points continued**

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**Statement 1.** (M.Kapovich-B. Kleiner-B.Leeb) Let \((X, \text{dist})\) be a geodesic metric space. The asymptotic cone \(\mathcal{C} = \text{Con}^\omega(X, (o_n), (d_n))\) has cut points if \(X\) contains a sequence of geodesics \(g_n, n = 1, 2, \ldots\) with \(|g_n| = O(d_n)\), \(\text{dist}(g_n, o_n) = O(d_n)\), and superlinear divergence.

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**Statement 3** Let \(\mathbb{F} = (X_n, \mathcal{P}_n)\) be a sequence of homogeneous unbounded tree-graded metric spaces with observation points \(o_n\). Let \(\omega\) be an ultrafilter. Then the ultralimit \(\lim^\omega (\mathbb{F}, (o_n))\) has a tree-graded structure with a non-trivial transversal tree at every point.
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**Proposition.** (M. Kapovich-B.Kleiner-B.Leeb) A CAT(0) group $G$ acting on (CAT(0)) $X$ does not have cut points in its asymptotic cones iff every bi-infinite geodesic bounds a half-plane.
Examples

Groups and other metric spaces whose asymptotic cones have cut-points:
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- relatively hyperbolic groups and metrically relatively hyperbolic spaces (Drutu, Osin, Sapir);
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(Olshanskii - S.) There exists a f.g. group such that one asymptotic cone has cut points and another one does not.
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Actions on tree-graded spaces

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Our main result shows that a group acting “nicely” on a tree-graded space also acts “nicely” on an $\mathbb{R}$-tree.
Stabilizers

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- $\mathcal{C}_3(G)$ is the set of stabilizers of triples of points of $\mathbb{F}$ neither from the same piece nor on the same transversal geodesic.
The main result

**Theorem** Let $G$ be a finitely generated group acting on a tree-graded space $(\mathbb{F}, \mathcal{P})$. Suppose that the following hold:
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(I) The group $G$ acts by isometries on a complete $\mathbb{R}$-tree non-trivially, with stabilizers of non-trivial arcs in $C_2(G)$, and with stabilizers of non-trivial tripods in $C_3(G)$. 
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(IV) The group $G$ acts on a complete $\mathbb{R}$-tree by isometries, non-trivially, stabilizers of non-trivial arcs are locally inside $C_1(G)$-by-Abelian subgroups, and stabilizers of tripods are locally inside subgroups in $C_1(G)$.
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**Theorem** Let $G$ be a finitely presented group acting on a tree-graded space $(F, P)$. Suppose that the following hold:

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Splitting

**Theorem (V. Guirardel)** Let $G$ be a finitely generated group and let $T$ be a real tree on which $G$ acts minimally.
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3. $T$ is a line and $G$ has a subgroup of index at most 2 that is an extension of the kernel of that action by a finitely generated free Abelian group.
When an action satisfies Guirardel’s condition?

**Statement.** Let $G$ be a finitely generated group acting on an $\mathbb{R}$-tree $T$ with finite of size at most $D$ tripod stabilizers,
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**Statement.** Let $G$ be a finitely generated group acting on an $\mathbb{R}$-tree $T$ with finite of size at most $D$ tripod stabilizers, and (finite of size at most $D$)-by-Abelian arc stabilizers,
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**Statement.** Let $G$ be a finitely generated group acting on an $\mathbb{R}$-tree $T$ with finite of size at most $D$ tripod stabilizers, and (finite of size at most $D$)-by-Abelian arc stabilizers, for some constant $D$. 
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Hence $D > |H \cap hHh^{-1}| = |H| > (D + 1)!$. 
Dahmani’s result

**Definition** Following Dahmani, we say that a homomorphism $\phi$ from a group $\Lambda$ into a relatively hyperbolic group $G$ has an *accidental parabolic* if either $\phi(\Lambda)$ is parabolic or
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**Definition** Following Dahmani, we say that a homomorphism $\phi$ from a group $\Lambda$ into a relatively hyperbolic group $G$ has an *accidental parabolic* if either $\phi(\Lambda)$ is parabolic or $\Lambda$ splits over a subgroup $C$ such that $\phi(C)$ is either parabolic or finite.
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**Definition** Following Dahmani, we say that a homomorphism $\phi$ from a group $\Lambda$ into a relatively hyperbolic group $G$ has an *accidental parabolic* if either $\phi(\Lambda)$ is parabolic or $\Lambda$ splits over a subgroup $C$ such that $\phi(C)$ is either parabolic or finite.

**Theorem (Dahmani)** If $\Lambda$ is finitely presented, and $G$ is relatively hyperbolic then there are finitely many subgroups of $G$, up to conjugacy, that are images of $\Lambda$ in $G$ by homomorphisms without accidental parabolics.
Homomorphisms into groups

Instead of homomorphic images, we consider the set of homomorphisms.
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Note that if a group $G$ splits over an Abelian subgroup $C$, say, $G = A \ast_C B$, then it typically has many outer automorphisms that are identity on $A$ and conjugate $B$ by elements of $C$. Hence we need to modify the definition of accidental parabolics as follows.
Definition. A homomorphism $\phi: \Lambda \rightarrow G$ has a weakly accidental parabolic if either $\phi(\Lambda)$ is parabolic or
**Definition.** A homomorphism $\phi: \Lambda \to G$ has a *weakly accidental parabolic* if either $\phi(\Lambda)$ is parabolic or $\Lambda$ splits over a subgroup $C$ such that $\phi(C)$ is either virtually cyclic or parabolic.
Weakly accidental parabolics

**Definition.** A homomorphism $\phi: \Lambda \rightarrow G$ has a *weakly accidental parabolic* if either $\phi(\Lambda)$ is parabolic or $\Lambda$ splits over a subgroup $C$ such that $\phi(C)$ is either virtually cyclic or parabolic.

**Theorem** Let $\Lambda$ be a finitely generated group, $G$ be a relatively hyperbolic group and parabolic subgroups are small (no free non-Abelian subgroups).
Weakly accidental parabolics

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**Theorem** Let $\Lambda$ be a finitely generated group, $G$ be a relatively hyperbolic group and parabolic subgroups are small (no free non-Abelian subgroups).

Then the number of pairwise non-conjugate in $G$ injective homomorphisms $\Lambda \to G$ without weakly accidental parabolics is finite.
Out($G$)

Relatively hyperbolic groups with infinite Out($G$) and non-co-Hopf relatively hyperbolic groups have been studied extensively (Paulin, Rips-Sela, T.Delzant-L.Potyagailo, D. Groves and I. Belegradek - A. Szczepański.)
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- $G$ can be represented as a non-trivial amalgamated product or HNN extension with one of the vertex groups a maximal parabolic subgroup of $G$. 

$\text{Out}(G)$
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Given an action of $G$ on $\mathcal{F}$, we need to extract an action of $G$ on a tree.
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The most natural tree, associated with any tree-graded space is essentially the union of all transversal trees, and can be described as a certain factor-space $F/\approx$. The action of $G$ on $F$ induces an action of $G$ on $F/\approx$. 

Squeezing apples
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Given an action of $G$ on $F$, we need to extract an action of $G$ on a tree.

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Squeezing apples
An example

An example of a non-trivial tree-graded structure: $X$ is a unit interval, pieces are “mid thirds” used to obtain the Cantor set, and single points.
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Note that pieces do not intersect.
Case A.

Suppose that the action of $G$ on $T = \mathbb{F}/ \approx$ is non-trivial.
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Suppose that the action of $G$ on $T = \mathbb{F}/\approx$ is non-trivial. Then the stabilizer of an arc in $T$ is from $\mathcal{C}_2$.

Indeed, every arc in $T$ contains an arc from a transverse tree of $\mathbb{F}$. 
Case A.

Suppose that the action of $G$ on $T = \mathbb{F}/\sim$ is non-trivial. Then the stabilizer of an arc in $T$ is from $C_2$.

Indeed, every arc in $T$ contains an arc from a transverse tree of $\mathbb{F}$.

Thus in this case $G$ acts non-trivially on an $\mathbb{R}$-tree with arc stabilizers from $C_2$. 
Case B.

Suppose that $G$ fixes a point in $T$. 
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The corresponding $\approx$-class is a union of pieces and is a tree-graded space $(R, \mathcal{R})$ with trivial transversal trees. $G$ acts on $R$. 
Transfinite sequence of tree-graded structures

Let $(\mathcal{F}, \mathcal{P})$ be a tree-graded space.
Transfinite sequence of tree-graded structures

Let $(F, P)$ be a tree-graded space.

We define $a \sim b$ iff $[a, b]$ is covered by finitely many pieces.
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A transfinite sequence of tree-graded structures

We have a sequence:

\[ \mathcal{P}_0 = \mathcal{R} < \mathcal{P}_1 = \mathcal{P}_0' < \mathcal{P}_2 = \mathcal{P}_1' \ldots \]
A transfinite sequence of tree-graded structures

We have a sequence:

\[ P_0 = R < P_1 = P_0' < P_2 = P_1' \ldots \]

It must stabilize at \( P_\alpha \).
Case B1.

$G$ fixes a piece in $\mathcal{P}_\alpha$. 
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Consider minimal $\delta$ such that $G$ fixes a piece in $P_\delta$. 
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We prove that $\delta$ is not a limit cardinal.
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$G$ fixes a piece in $\mathcal{P}_\alpha$.

Consider minimal $\delta$ such that $G$ fixes a piece in $\mathcal{P}_\delta$.

We prove that $\delta$ is not a limit cardinal.

Then we define a simplicial tree having pieces of $\mathcal{P}_{\delta - 1}$ and intersections of these pieces as vertices, and edges connecting a piece and a vertex inside it.
The stabilizers of edges depend on whether $\delta = 1$ or not.
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If $\delta = 1$, the stabilizers of edges are inside stabilizers of pieces in $\mathcal{P}$. 
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If $\delta = 1$, the stabilizers of edges are inside stabilizers of pieces in $\mathcal{P}$.

In case $\delta > 1$, the edge stabilizers are in $\mathcal{C}_1$. 
Case B2.

$G$ does not fix a point in $\mathcal{P}_\alpha$. 
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$G$ does not fix a point in $\mathcal{P}_\alpha$.

Then $G$ acts on the set $X$ of $\mathcal{P}_\alpha$-pieces.
Case B2.

$G$ does not fix a point in $P_{\alpha}$.

Then $G$ acts on the set $X$ of $P_{\alpha}$-pieces.

We define the structure of a pre-tree (Bowditch) on $X$. 
**Definition** A *pretree* is a set equipped with a ternary *betweenness* relation $xyz$ satisfying the following conditions:
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**Pretrees**

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Pretrees

Definition A pretree is a set equipped with a ternary betweenness relation \(xyz\) satisfying the following conditions:

- (PT0) \((\forall x, y)(\neg xyx)\).
- (PT1) \(xzy \iff yzx\).
- (PT2) \((\forall x, y, z)(\neg (xyz \land xzy))\).
- (PT3) \(xzy\) and \(z \neq w\) then \((xzw \lor yzw)\).
If $x, y, z$ are pieces of $\mathcal{P}_\alpha$ then we say $xyz$ iff there exists a geodesic in $\mathbb{F}$ starting in $x$, ending in $z$ and crossing $y$. 
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We apply a version of Levitt’s theorem and complete the proof.