Words and their meaning

Syntax and semantics in algebra

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By words we learn thoughts,
and by thoughts we learn life.
(Jean Baptiste Girard)

The Novikov-Adian theorem

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Theorem.(Novikov-Adian, Olshanskii) The 2-generated free Burnside group $B_{2,n}$ is infinite for sufficiently large odd n (say, $n > 10^{10}$).

The basic rough idea

Order the cyclically reduced words in the free group $F_2 = \langle\, a, b \,\rangle$: $u_1 < u_2 < \dots$. Consider the following sequence of groups G_i with group presentations

$$\mathcal{PB}_i = \langle\, a, b \mid C_1^n = 1, C_2^n = 1, \dots, C_i^n = 1 \,\rangle$$

where G_0 is the free group F_2, and C_i is the smallest word which has infinite order in G_{i-1}, for every $i \geq 1$.

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The theorem implies that $B_{2,n}$ is infinite.

j-pairs

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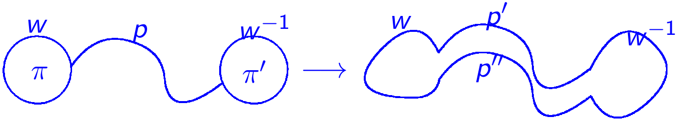
We can remove j-pairs reducing the type:



We call a van Kampen or annular diagram of rank i *reduced* if it does no contain j-pairs for any $j \leq i$.

The crown of lemmas

The proof consists of several lemmas proved by a simultaneous induction on the type of a diagram over $\mathcal{PB}$. It means that proving every lemma, we can assume that all other lemmas are already proved for diagrams of smaller type.

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In the proof, we often use the phrase that some quantity a (length of a path, or weight of a cell, etc.) is "much smaller" than the other quantity b. This usually mean that $a \leq \mu b$ for some very small parameter μ.

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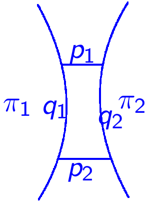
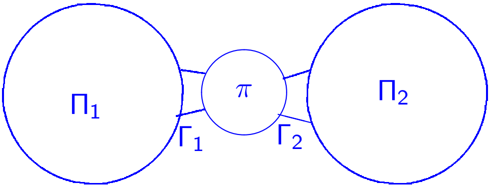


Bonds



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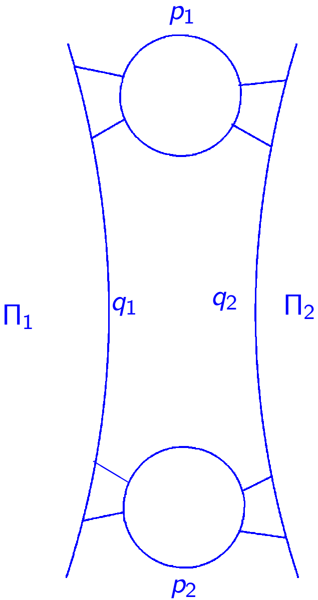
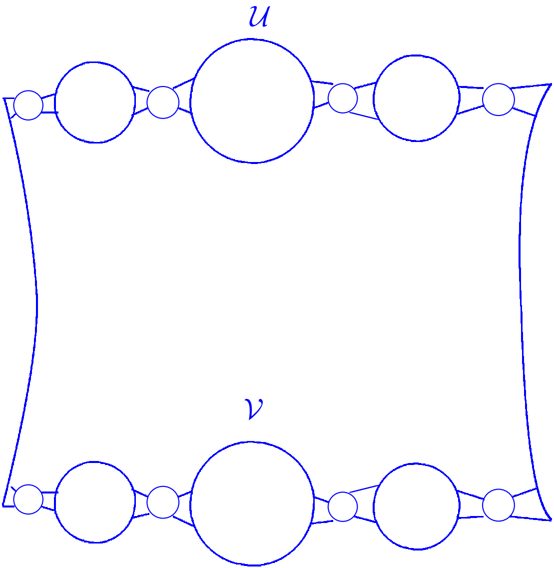
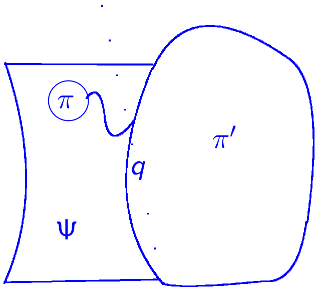


Bands of bonds



Boundary arcs: smooth, almost geodesic, compatible with a cell



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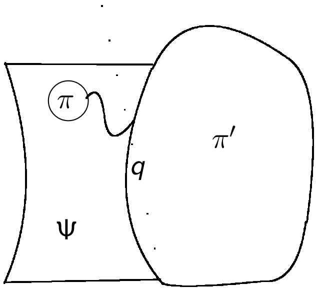

Smooth: no compatible cells, **geodesic**: cannot be shortened by homotopy inside the diagram.

A good system of contiguity subdiagrams

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Good: A system of contiguity subdiagrams which covers all edges and has minimal possible number of contiguity subdiagrams . Let Σ be a good collection of contiguity subdiagrams. Then all cells are divided into special (the main cells of the bands of bonds), concealed (inside the contiguity subdiagrams), and ordinary)

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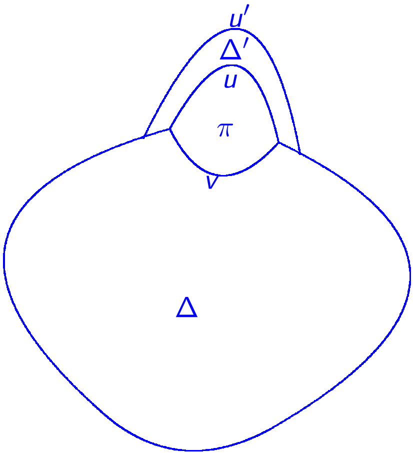

Weights

We assume that every diagram is *normal*. This means that the rank of a contiguity subdiagram is always less than the ranks of the cells or boundary arcs which this subdiagram connects.

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Use weights instead of lengths. The weight of an edge from $\partial(\pi)$ is $|\partial(\pi)|^{-1/3}$, the weight of π is $|\partial(\pi)|^{2/3}$. Bigger cells weight more, edges of bigger cells weight less.

A lemma about weighted graphs

Lemma. Let S be a circle with at most distinguished points $O_1, O_2, \ldots, O_l$ $(l \leq 4)$.

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$\nu(e) \leq a \min\{\nu(e_-), \nu(e_+)\}$

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Three lemmas about contiguity subdiagrams

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Lemma 3 The contiguity degree of π to Π cannot exceed a certain parameter α which is only a little bigger than $\frac{1}{2}$.

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Thus $S < \delta(S + O)$. Hence $S = o(O)$.

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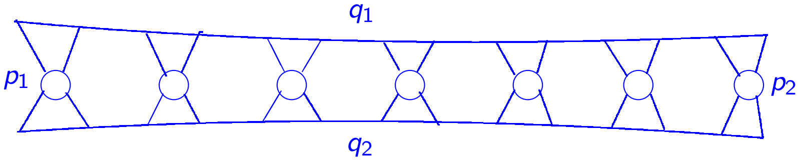
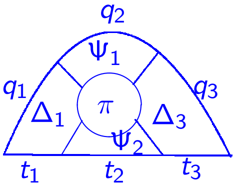


Why are smooth arcs almost geodesic?



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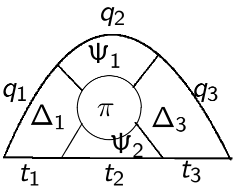

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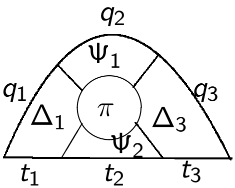

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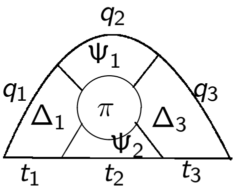

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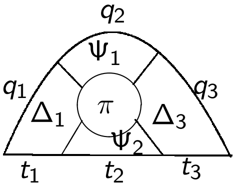

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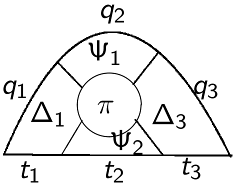

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This implies that the length of q is almost the same as the length of t as required.

More results obtained by cutting

If A and C are *simple words* in G_i, i.e. not conjugated in G_{i-1} to powers of shorter words, and A is conjugated to a power of C in G_i, $A = X^{-1}C^lX$, then $l = \pm 1$.

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This, in particular, implies that the group given by the presentation $\mathcal{PB}$ is indeed a group of exponent n.

Why do diagrams with small perimeters have small ranks?

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Let Δ be a reduced diagram with boundary q. Let π be a cell from Δ and Δ' be the annular diagram obtained by removing π from Δ. The boundary of π is smooth because Δ is a reduced diagram. Hence by the annular version of almost geodesicity, the length of $\partial(\pi)$ cannot be much bigger than $|q|$. Thus the rank of π cannot be large also.

Short cuts in annular diagrams

Lemma. Let Δ be a reduced annular diagram over $\mathcal{PB}$ with boundary components p, q. Then there are vertices o_1 in p and o_2 in q, and a path s connecting o_1, o_2 such that $|s|$ is much smaller than $|p| + |q|$ (say, $|s| < \frac{1}{100}(|p| + |q|)$).

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This is a standard fact about hyperbolic groups.

Fine-Wilf

We need to show that if one long side of a contiguity subdiagram has many periods then the other long side cannot contain more than 1 and a little bit of a period.

Fine-Wilf

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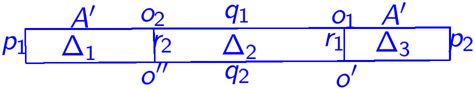
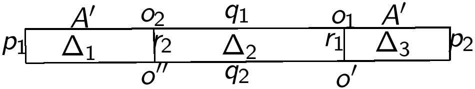
Lemma AA. Suppose that Δ is a diagram of rank i with $\partial(\Delta) = p_1 q_1 p_2^{-1} q_2^{-1}$ where p_1 and p_2 are very short (just how short p_i should be will be clear from the proof) and the labels u_1, u_2 of q_1 and q_2 are periodic words with period A which is simple in G_i, and $|u_1|$ contains large enough number of periods. Then the boundary arcs q_1 and q_2 are compatible.

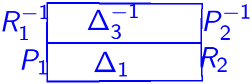
AA→ AC



AA→ AC

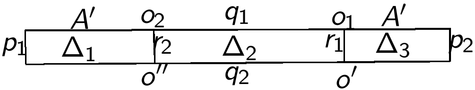


A'
o_2
q_1
o_1
A'
p_1
Δ_1
r_2
Δ_2
r_1
Δ_3
p_2
o''
q_2
o'

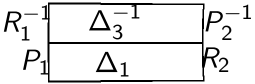


R_1^{-1}
Δ_3^{-1}
P_2^{-1}
P_1
Δ_1
R_2

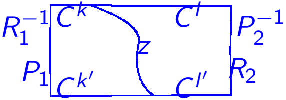
AA→ AC



A'
o_2
q_1
o_1
A'
p_1
Δ_1
r_2
Δ_2
r_1
Δ_3
p_2
o''
q_2
o'

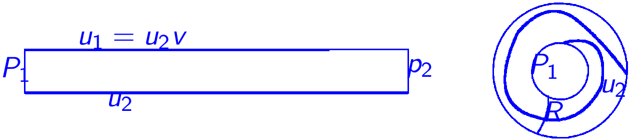


R_1^{-1}
Δ_3^{-1}
P_2^{-1}
P_1
Δ_1
R_2

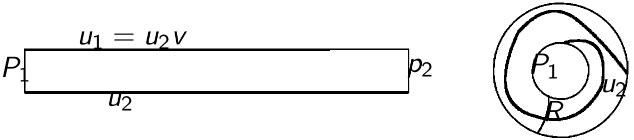


R_1^{-1}
C^k
C^l
P_2^{-1}
z
P_1
$C^{k'}$
$C^{l'}$
R_2

AC→ AA



AC→ AA



Use the fact that the fundamental group of a circle is $\mathbb{Z}$. So A^k is almost conjugate by a short word to a large power of a shorter word P_1 and we are in the situation of the AC-lemma. So AC $\rightarrow$ AA.