Cut-points in asymptotic cones of groups

Mark Sapir

With J. Behrstock, C. Druţu, S. Mozes, A.Olshanskii, D. Osin

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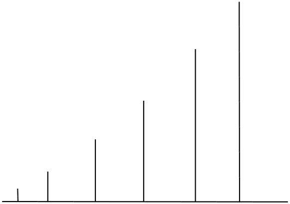

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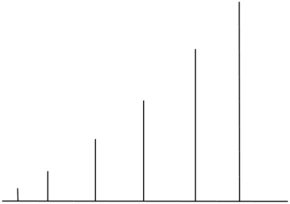

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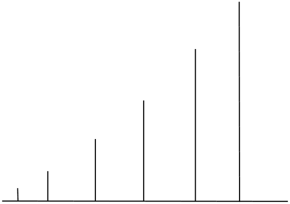

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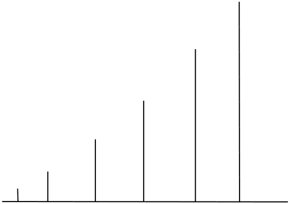

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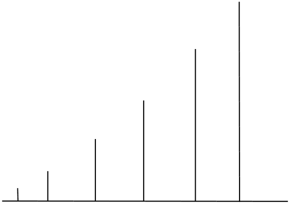

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Observation due to Bestvina and Paulin: if a group has many actions on a Gromov-hyperbolic metric space then it acts non-trivially (i.e. without a global fixed point) by isometries on the asymptotic cone of that space which is an $\mathbb{R}$-tree.

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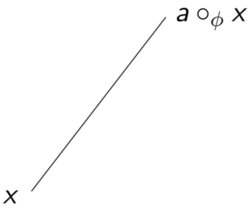

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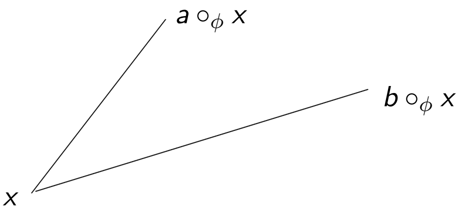

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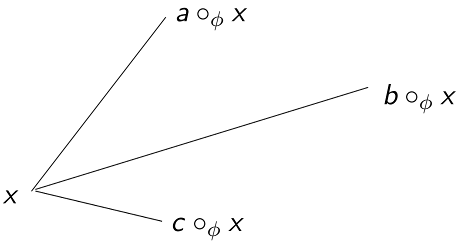

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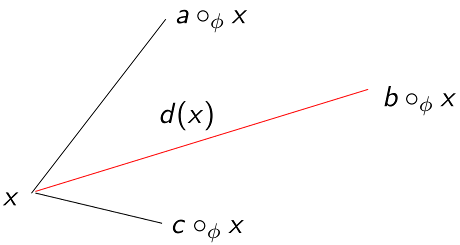

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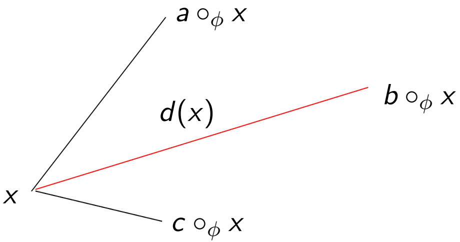

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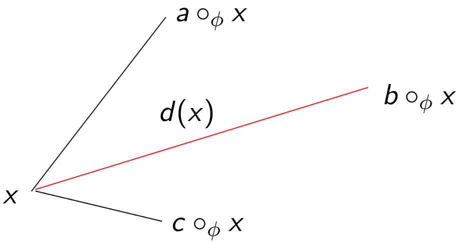

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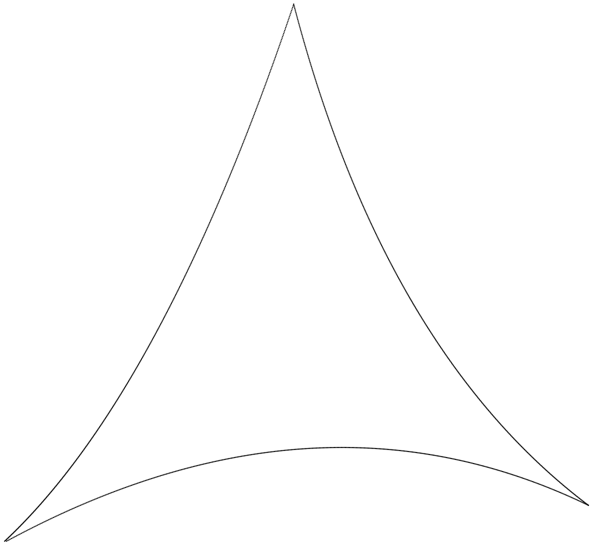
Indeed, triangles in X_ϕ become thinner and thinner and in the limit all geodesic triangles are tripods:



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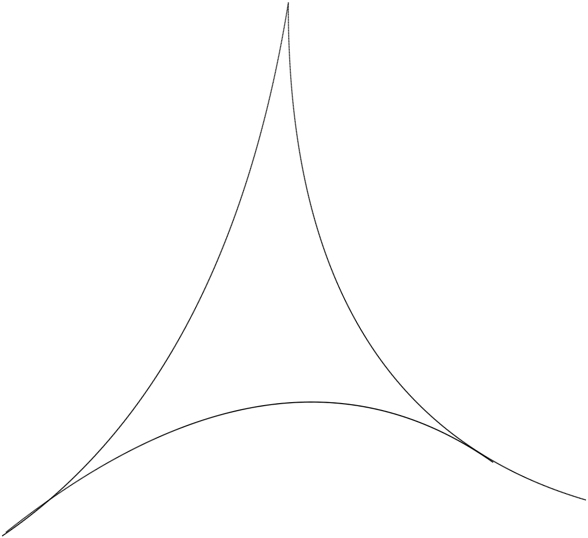

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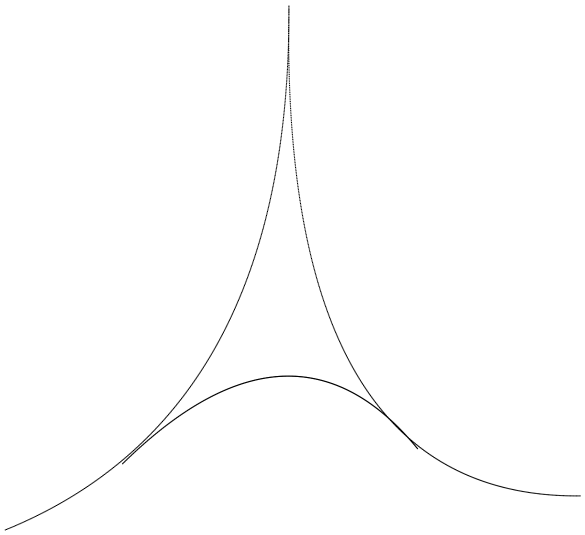

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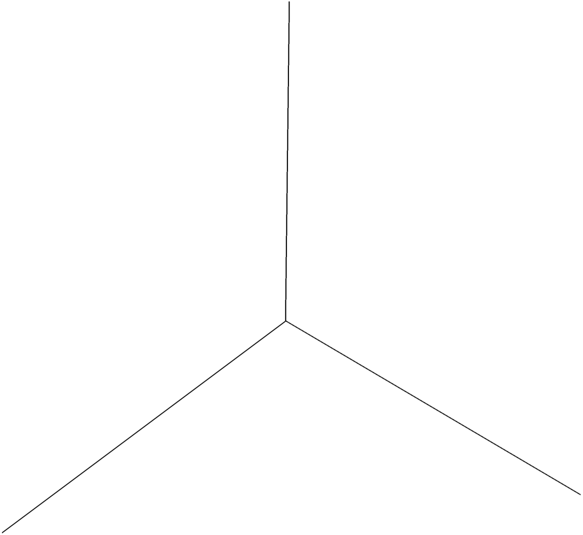

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The asymptotic cones of non-hyperbolic spaces need not be trees.

But in many cases they are tree-graded spaces. Recall the definition.

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Then we say that the space $\mathbb{F}$ is *tree-graded with respect to* $\mathcal{P}$.

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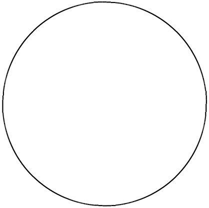

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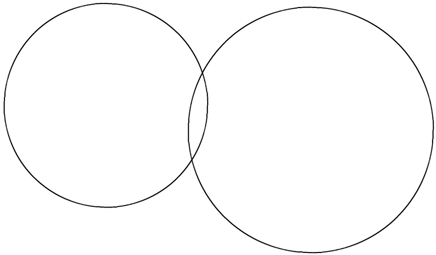

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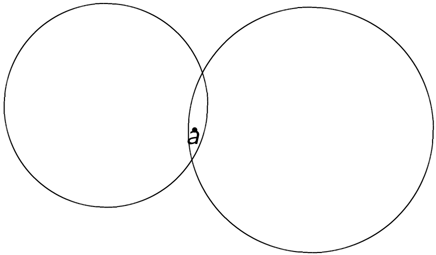

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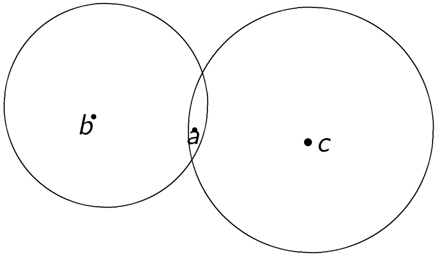

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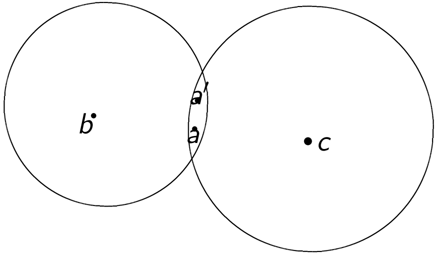

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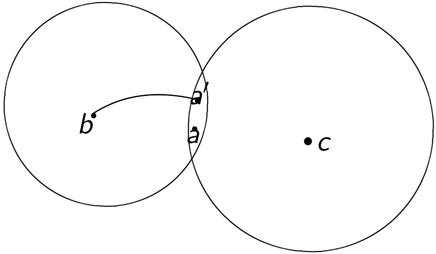

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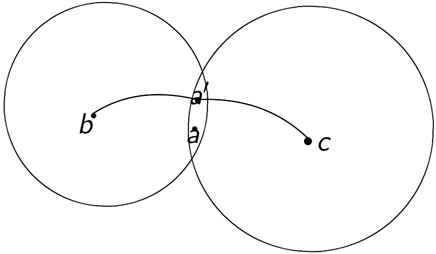

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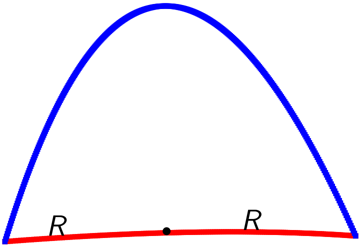

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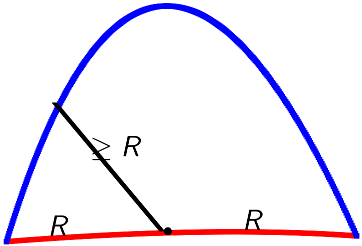

The length of the blue arc should be $> O(R)$.

Cut points and tree-graded structures

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Cut points and tree-graded structures

Recall that hyperbolicity $\equiv$superlinear divergence of any pair of geodesic rays with common origin.

Transversal trees

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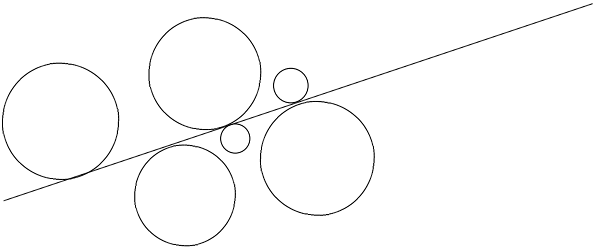

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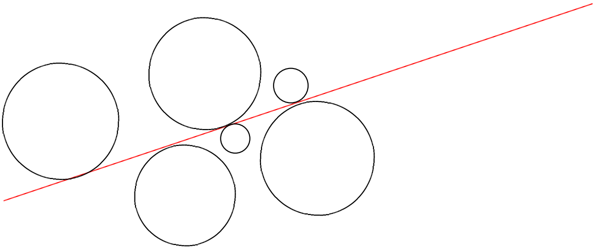

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The line is a transversal tree, the other transversal trees are points on the circles.

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Proposition. Let X be a homogeneous geodesic metric space such that one of the asymptotic cones of X has a cut point.

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Actions on tree-graded spaces

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Our main result shows that a group acting “nicely” on a tree-graded space also acts “nicely” on an $\mathbb{R}$-tree.

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Question. Is there a f.g. (f.p.) amenable group with cut points in every a.c.?

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Theorem (Dahmani) If Λ is finitely presented, and G is relatively hyperbolic then there are finitely many subgroups of G, up to conjugacy, that are images of Λ in G by homomorphisms without accidental parabolics.

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Note that if a group G splits over an Abelian subgroup C, say, $G = A *_C B$, then it typically has many outer automorphisms that are identity on A and conjugate B by elements of C. Hence we need to modify the definition of accidental parabolics as follows.

Weakly accidental parabolics

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Theorem Let Λ be a finitely generated group, G be a relatively hyperbolic group and parabolic subgroups are small (no free non-Abelian subgroups).

Then the number of pairwise non-conjugate in G injective homomorphisms $\Lambda \to G$ without weakly accidental parabolics is finite.

$\mathrm{Out}(G)$

Relatively hyperbolic groups with infinite $\mathrm{Out}(G)$ and non-co-Hopf relatively hyperbolic groups have been studied extensively (Paulin, Rips-Sela, T.Delzant-L.Potyagailo, D. Groves and I. Belegradek - A. Szczepański.)

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If $\mathrm{Out}(G)$ is infinite then one of the followings cases occurs.

- G splits over a virtually cyclic subgroup;
- G splits over a parabolic (finite of uniformly bounded size)-by-Abelian-by-(virtually cyclic) subgroup;

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If $\mathrm{Out}(G)$ is infinite then one of the followings cases occurs.

- G splits over a virtually cyclic subgroup;
- G splits over a parabolic (finite of uniformly bounded size)-by-Abelian-by-(virtually cyclic) subgroup;
- G can be represented as a non-trivial amalgamated product or HNN extension with one of the vertex groups a maximal parabolic subgroup of G.

co-Hopfian groups

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Theorem Suppose that a relatively hyperbolic group G is not co-Hopfian.

Let ϕ be an injective but not surjective homomorphism $G \to G$.

Then one of the following holds:

- $\phi^k(G)$ is parabolic for some k.
- G splits over a parabolic or virtually cyclic subgroup.