Cut-points in asymptotic cones of groups

Mark Sapir

With J. Behrstock, C. Druțu, S. Mozes, A.Olshanskii, D. Osin
Asymptotic cones

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Druțu, S. a f.g. group with continuum a.c.
Observation due to Bestvina and Paulin: if a group has many actions on a Gromov-hyperbolic metric space then it acts non-trivially (i.e. without a global fixed point) by isometries on the asymptotic cone of that space which is an $\mathbb{R}$-tree.
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Then we can divide the metric in $X$ by $d_\phi$, obtaining $X_\phi$, $\phi: \Lambda \to G$. The $\mathbb{R}$-tree is the limit $\text{Con}(X, (d_\phi), (x_\phi))$. 

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But in many cases they are tree-graded spaces. Recall the definition.
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Tree-graded spaces

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Then we say that the space $\mathbb{F}$ is *tree-graded with respect to* $\mathcal{P}$. 
Cut points and tree-graded structures

Note (Druțu, S.). Any complete geodesic metric space with cut-points has non-trivial canonical tree-graded structure:
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\begin{array}{c}
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  b^* \quad a \quad c
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The length of the blue arc should be $> O(R)$. 

![Diagram showing a blue arc with length greater than $O(R)$]
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Recall that hyperbolicity $\equiv$
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Transversal trees

**Definition.** For every point $x$ in a tree-graded space $(F, P)$, the union of geodesics $[x, y]$ intersecting every piece by at most one point is an $\mathbb{R}$-tree called a *transversal* tree of $F$. 
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The geodesics $[x, y]$ from transversal trees are called *transversal geodesics*. 
Transversal trees, an example
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A tree-graded space. Pieces are the circles and the points on the line.
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The line is a transversal tree, the other transversal trees are points on the circles.
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Observation. (D+S) A bi-infinite geodesic in the Cayley graph is Morse iff its limit in every asymptotic cone is a transversal geodesic.
Actions on tree-graded spaces

Thus it is important to study actions of groups on tree-graded spaces.
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Our main result shows that a group acting “nicely” on a tree-graded space also acts “nicely” on an $\mathbb{R}$-tree.
Stabilizers

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- $\mathcal{C}_3(G)$ is the set of stabilizers of triples of points of $\mathbb{F}$ neither from the same piece nor on the same transversal geodesic.
The main result

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**Theorem** Let \( G \) be a finitely generated group acting on a tree-graded space \((\mathbb{F}, \mathcal{P})\). Suppose that the following hold:

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(IV) The group \( G \) acts on a complete \( \mathbb{R} \)-tree by isometries, non-trivially, stabilizers of non-trivial arcs are locally inside \( C_1(G) \)-by-Abelian subgroups, and stabilizers of tripods are locally inside subgroups in \( C_1(G) \).
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**Theorem** Let $G$ be a finitely presented group acting on a tree-graded space $(\mathbb{F}, \mathcal{P})$. Suppose that the following hold:

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Examples (no cut points)

Groups and other metric spaces whose asymptotic cones do not have cut-points:
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**Question** What about non-classical Lie groups?
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Question. Is there a f.g. (f.p.) amenable group with cut points in every a.c.?
Dahmani’s result

**Definition** Following Dahmani, we say that a homomorphism $\phi$ from a group $\Lambda$ into a relatively hyperbolic group $G$ has an *accidental parabolic* if either $\phi(\Lambda)$ is parabolic or
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**Definition** Following Dahmani, we say that a homomorphism $\phi$ from a group $\Lambda$ into a relatively hyperbolic group $G$ has an *accidental parabolic* if either $\phi(\Lambda)$ is parabolic or $\Lambda$ splits over a subgroup $C$ such that $\phi(C)$ is either parabolic or finite.

**Theorem (Dahmani)** If $\Lambda$ is finitely presented, and $G$ is relatively hyperbolic then there are finitely many subgroups of $G$, up to conjugacy, that are images of $\Lambda$ in $G$ by homomorphisms without accidental parabolics.
Homomorphisms into groups

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Note that if a group $G$ splits over an Abelian subgroup $C$, say, $G = A \ast_C B$, then it typically has many outer automorphisms that are identity on $A$ and conjugate $B$ by elements of $C$. Hence we need to modify the definition of accidental parabolics as follows.
Weakly accidental parabolics

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**Theorem** Let $\Lambda$ be a finitely generated group, $G$ be a relatively hyperbolic group and parabolic subgroups are small (no free non-Abelian subgroups).
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**Theorem** Let $\Lambda$ be a finitely generated group, $G$ be a relatively hyperbolic group and parabolic subgroups are small (no free non-Abelian subgroups).

Then the number of pairwise non-conjugate in $G$ injective homomorphisms $\Lambda \to G$ without weakly accidental parabolics is finite.
Out$(G)$

Relatively hyperbolic groups with infinite $\text{Out}(G)$ and non-co-Hopf relatively hyperbolic groups have been studied extensively (Paulin, Rips-Sela, T.Delzant-L.Potyagailo, D. Groves and I. Belegradek - A. Szczepański.)
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co-Hopfian groups

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