Hilbert space compression of groups

The definition: compression

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The Hilbert space compression of a space is a q.i. invariant.

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Not a linear order. So we cannot talk about *the maximal* compression function.

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  (Amenability - for the equivariant compression.)

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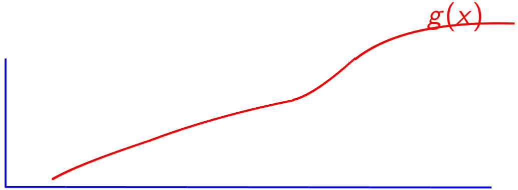

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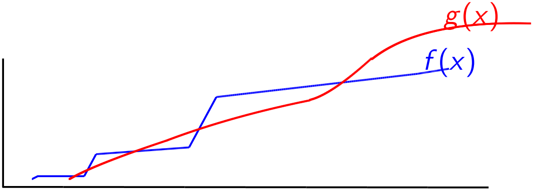

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Problem. Is there a non-virtually cyclic group with better compression gap than F_n?

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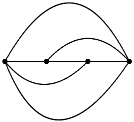
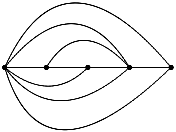


x_0



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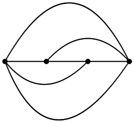

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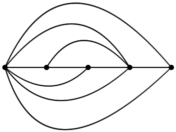

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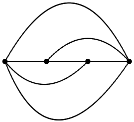

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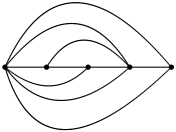

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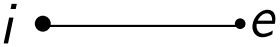

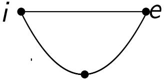

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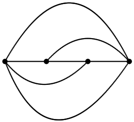

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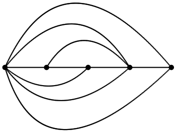

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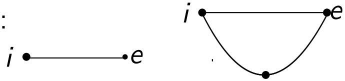

Operations:

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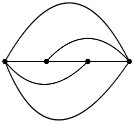

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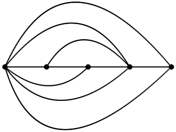
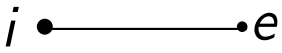
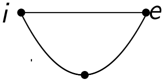


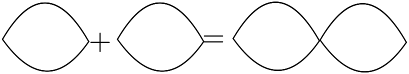
x_1

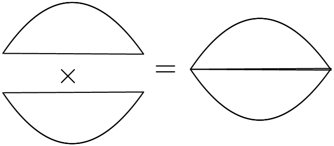
Elementary diagrams:





Operations:





Thompson group as a diagram group

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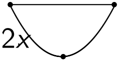

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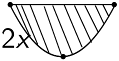

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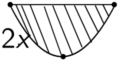

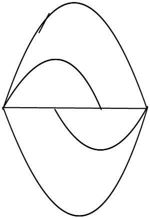

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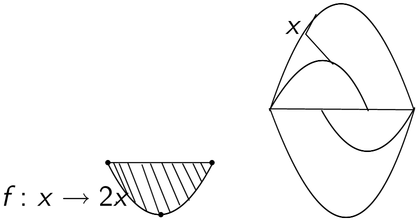

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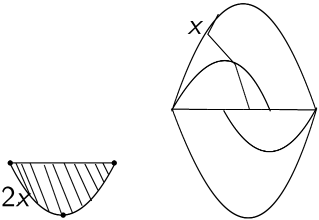

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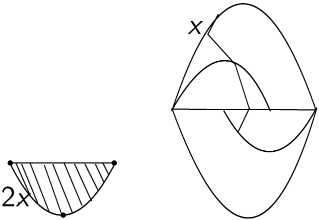

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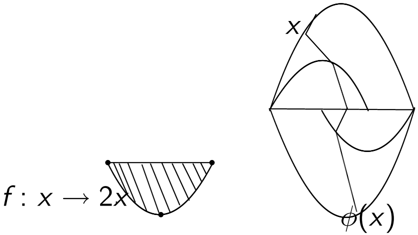

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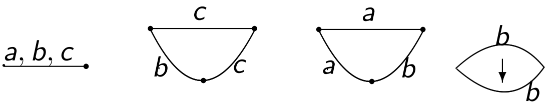

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Idea of the proof. Free group acts on a tree, Thompson group (and any other diagram group) acts of a 2-tree.

2-trees

How to build the tree (Cayley graph of F_3):

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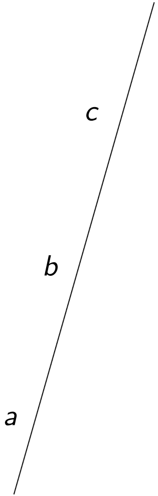

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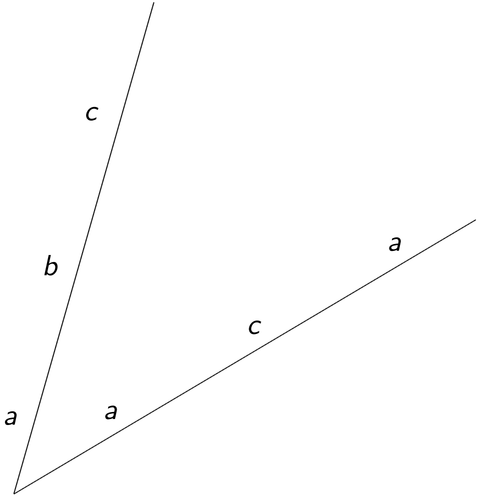

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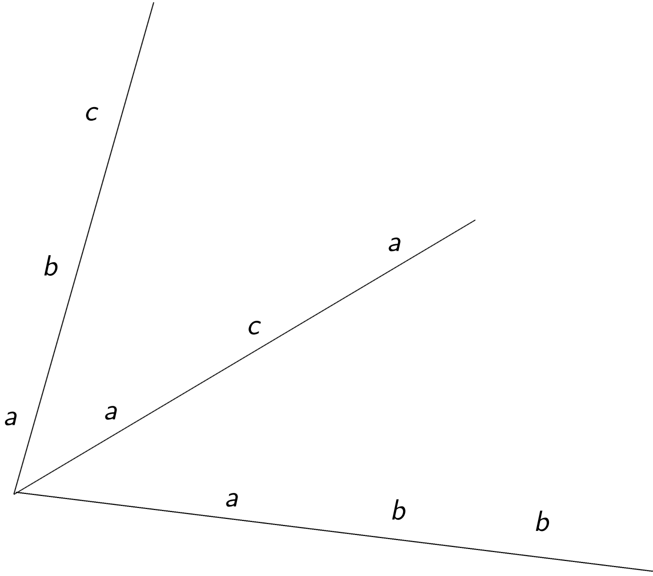

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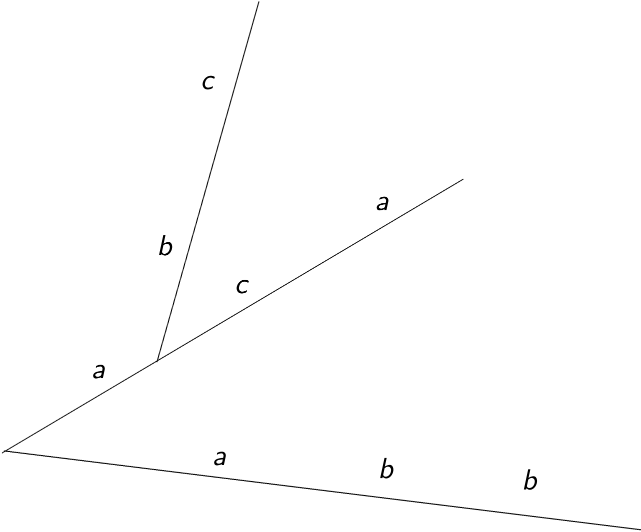

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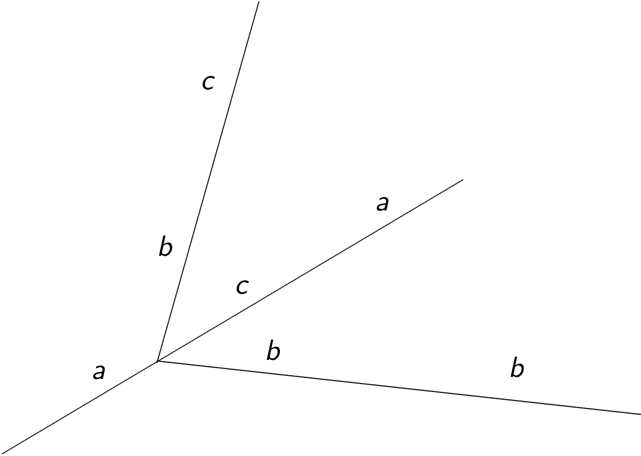

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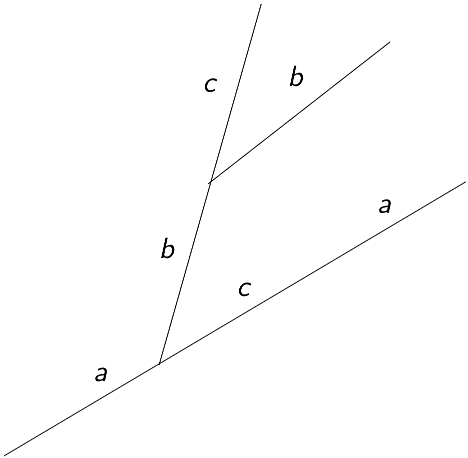

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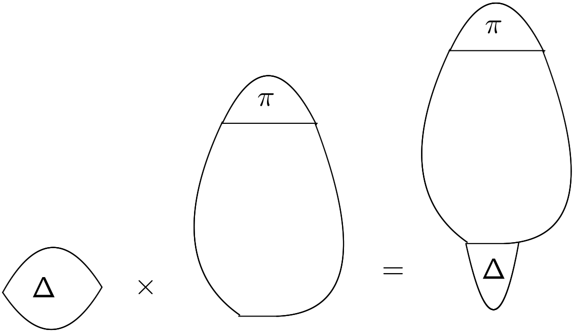

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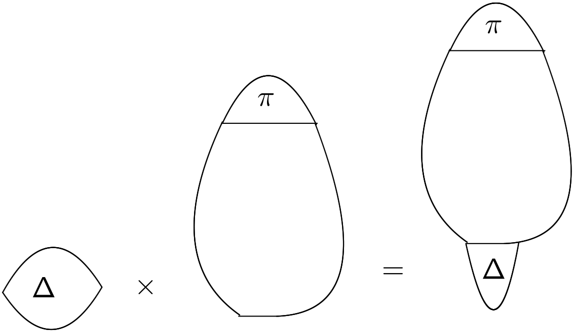

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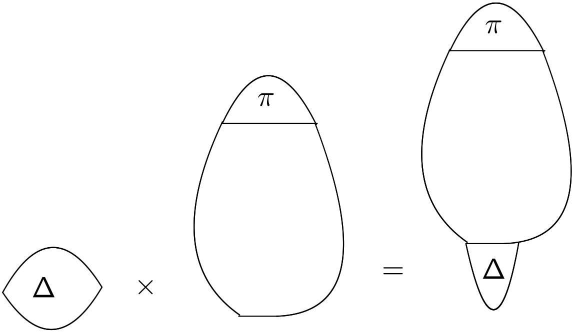

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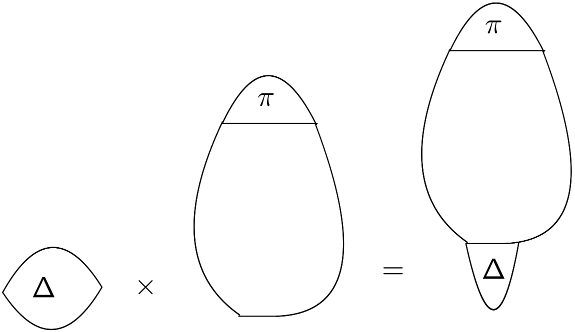

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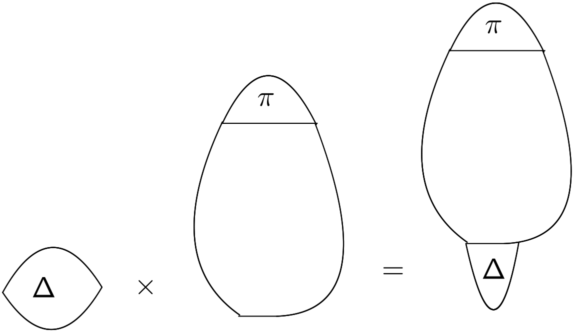

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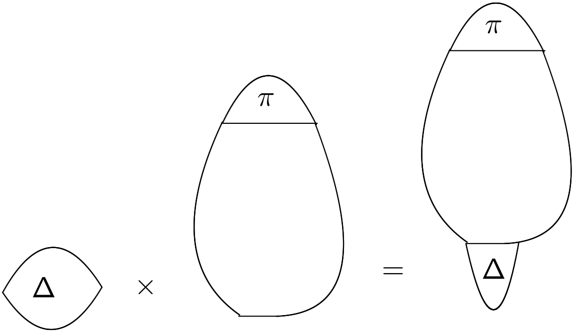

Let $H = l^2$(set of 2-cells of the 2-tree). Every diagram Δ has unique image on the 2-tree. Map it to the sum of the cells of the image. The compression function is $\sqrt{x}$. The equivariant compression of the embedding $= \frac{1}{2}$ because of the Burillo property.

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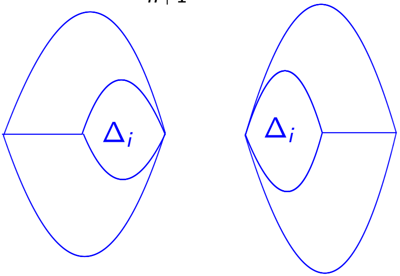

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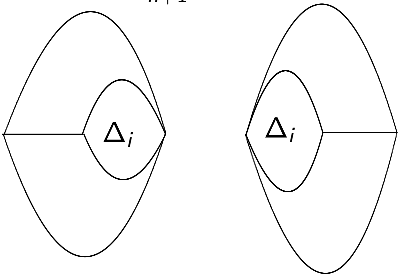

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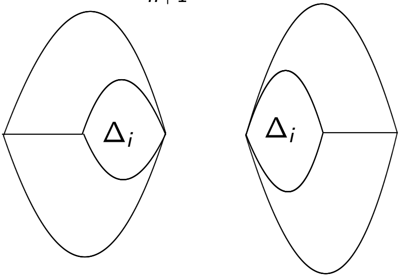

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The problem

Problem. Is it true that a compression function of some embedding of F into a Hilbert space is $\gg \sqrt{x}$?

Wreath products

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Problem. Is there an amenable group with compression 0?