# Algorithmic and asymptotic properties of groups

Mark Sapir

# Group presentations

The object of our study - finitely generated groups given by presentations $\langle a_1, \ldots, a_n \mid r_1, r_2, \ldots \rangle$,

# Group presentations

The object of our study - finitely generated groups given by presentations $\langle a_1, \ldots, a_n \mid r_1, r_2, \ldots \rangle$, where $r_i$ is a word in $a_1, \ldots, a_n$.

# Group presentations

The object of our study - finitely generated groups given by presentations $\langle a_1, \dots, a_n \mid r_1, r_2, \dots \rangle$, where $r_i$ is a word in $a_1, \dots, a_n$.
That is groups generated by $a_1, \dots, a_n$

# Group presentations

The object of our study - finitely generated groups given by presentations $\langle a_1, \ldots, a_n \mid r_1, r_2, \ldots \rangle$, where $r_i$ is a word in $a_1, \ldots, a_n$. That is groups generated by $a_1, \ldots, a_n$ with relations $r_1 = 1, r_2 = 1, \ldots$ imposed.

# Group presentations

The object of our study - finitely generated groups given by presentations $\langle a_1, ..., a_n \mid r_1, r_2, ... \rangle$, where $r_i$ is a word in $a_1, ..., a_n$. That is groups generated by $a_1, ..., a_n$ with relations $r_1 = 1, r_2 = 1, ...$ imposed.

For example, the free Burnside group of exponent $n$ with two generators is given by the presentation

# Group presentations

The object of our study - finitely generated groups given by presentations $\langle a_1, ..., a_n \mid r_1, r_2, ...\rangle$, where $r_i$ is a word in $a_1, ..., a_n$. That is groups generated by $a_1, ..., a_n$ with relations $r_1 = 1, r_2 = 1, ...$ imposed.

For example, the free Burnside group of exponent $n$ with two generators is given by the presentation

$$\langle a, b \mid u^n = 1\rangle$$

for all words $u$ in the alphabet $a, b$.

# Group presentations

The object of our study - finitely generated groups given by presentations $\langle a_1, ..., a_n \mid r_1, r_2, ...\rangle$, where $r_i$ is a word in $a_1, ..., a_n$. That is groups generated by $a_1, ..., a_n$ with relations $r_1 = 1, r_2 = 1, ...$ imposed.

For example, the free Burnside group of exponent $n$ with two generators is given by the presentation

$$\langle a, b \mid u^n = 1\rangle$$

for all words $u$ in the alphabet $a, b$.

The fundamental group of the orientable surface of genus $n$ is given by the presentation

# Group presentations

The object of our study - finitely generated groups given by presentations $\langle a_1, ..., a_n \mid r_1, r_2, ...\rangle$, where $r_i$ is a word in $a_1, ..., a_n$. That is groups generated by $a_1, ..., a_n$ with relations $r_1 = 1, r_2 = 1, ...$ imposed.

For example, the free Burnside group of exponent $n$ with two generators is given by the presentation

$$\langle a, b \mid u^n = 1\rangle$$

for all words $u$ in the alphabet $a, b$.

The fundamental group of the orientable surface of genus $n$ is given by the presentation

$$\langle a_1, b_1, ..., a_n, b_n \mid [a_1, b_1]...[a_n, b_n] = 1\rangle.$$

# Classical results

**Theorem.** (Boone-Novikov's solution of Dehn's problem)

# Classical results

**Theorem.** (Boone-Novikov's solution of Dehn's problem) There exists a finitely presented group with undecidable word problem.

# Classical results

**Theorem.** (Boone-Novikov's solution of Dehn's problem) There exists a finitely presented group with undecidable word problem.

**Theorem.** (Higman)

# Classical results

**Theorem.** (Boone-Novikov's solution of Dehn's problem) There exists a finitely presented group with undecidable word problem.

**Theorem.** (Higman) A group has r.e. word problem iff

# Classical results

**Theorem.** (Boone-Novikov's solution of Dehn's problem) There exists a finitely presented group with undecidable word problem.

**Theorem.** (Higman) A group has r.e. word problem iff it is a subgroup of a f.p. group.

# Classical results

**Theorem.** (Boone-Novikov's solution of Dehn's problem) There exists a finitely presented group with undecidable word problem.

**Theorem.** (Higman) A group has r.e. word problem iff it is a subgroup of a f.p. group.

**Theorem.** (Adian-Novikov's solution of Burnside problem)

# Classical results

**Theorem.** (Boone-Novikov's solution of Dehn's problem) There exists a finitely presented group with undecidable word problem.

**Theorem.** (Higman) A group has r.e. word problem iff it is a subgroup of a f.p. group.

**Theorem.** (Adian-Novikov's solution of Burnside problem) The free Burnside group of exponent $n$ with at least two generators

# Classical results

**Theorem.** (Boone-Novikov's solution of Dehn's problem) There exists a finitely presented group with undecidable word problem.

**Theorem.** (Higman) A group has r.e. word problem iff it is a subgroup of a f.p. group.

**Theorem.** (Adian-Novikov's solution of Burnside problem) The free Burnside group of exponent $n$ with at least two generators is infinite for large enough odd $n$.

# Classical results

**Theorem.** (Boone-Novikov's solution of Dehn's problem) There exists a finitely presented group with undecidable word problem.

**Theorem.** (Higman) A group has r.e. word problem iff it is a subgroup of a f.p. group.

**Theorem.** (Adian-Novikov's solution of Burnside problem) The free Burnside group of exponent $n$ with at least two generators is infinite for large enough odd $n$.

**Theorem.** (Olshanskii's solution of Tarski's and von Neumann's problems)

# Classical results

**Theorem.** (Boone-Novikov's solution of Dehn's problem) There exists a finitely presented group with undecidable word problem.

**Theorem.** (Higman) A group has r.e. word problem iff it is a subgroup of a f.p. group.

**Theorem.** (Adian-Novikov's solution of Burnside problem) The free Burnside group of exponent $n$ with at least two generators is infinite for large enough odd $n$.

**Theorem.** (Olshanskii's solution of Tarski's and von Neumann's problems) There exists a non-Amenable group

# Classical results

**Theorem.** (Boone-Novikov's solution of Dehn's problem) There exists a finitely presented group with undecidable word problem.

**Theorem.** (Higman) A group has r.e. word problem iff it is a subgroup of a f.p. group.

**Theorem.** (Adian-Novikov's solution of Burnside problem) The free Burnside group of exponent $n$ with at least two generators is infinite for large enough odd $n$.

**Theorem.** (Olshanskii's solution of Tarski's and von Neumann's problems) There exists a non-Amenable group with all proper subgroups cyclic of the same prime order.

# Classical results

**Theorem.** (Boone-Novikov's solution of Dehn's problem) There exists a finitely presented group with undecidable word problem.

**Theorem.** (Higman) A group has r.e. word problem iff it is a subgroup of a f.p. group.

**Theorem.** (Adian-Novikov's solution of Burnside problem) The free Burnside group of exponent $n$ with at least two generators is infinite for large enough odd $n$.

**Theorem.** (Olshanskii's solution of Tarski's and von Neumann's problems) There exists a non-Amenable group with all proper subgroups cyclic of the same prime order.

**Theorem.** (Gromov's solution of Milnor's problem)

# Classical results

**Theorem.** (Boone-Novikov's solution of Dehn's problem) There exists a finitely presented group with undecidable word problem.

**Theorem.** (Higman) A group has r.e. word problem iff it is a subgroup of a f.p. group.

**Theorem.** (Adian-Novikov's solution of Burnside problem) The free Burnside group of exponent $n$ with at least two generators is infinite for large enough odd $n$.

**Theorem.** (Olshanskii's solution of Tarski's and von Neumann's problems) There exists a non-Amenable group with all proper subgroups cyclic of the same prime order.

**Theorem.** (Gromov's solution of Milnor's problem) Any group of polynomial growth

# Classical results

**Theorem.** (Boone-Novikov's solution of Dehn's problem) There exists a finitely presented group with undecidable word problem.

**Theorem.** (Higman) A group has r.e. word problem iff it is a subgroup of a f.p. group.

**Theorem.** (Adian-Novikov's solution of Burnside problem) The free Burnside group of exponent $n$ with at least two generators is infinite for large enough odd $n$.

**Theorem.** (Olshanskii's solution of Tarski's and von Neumann's problems) There exists a non-Amenable group with all proper subgroups cyclic of the same prime order.

**Theorem.** (Gromov's solution of Milnor's problem) Any group of polynomial growth has a nilpotent subgroup of finite index.

# Groups turning into machines

Groups $\longrightarrow$

# Groups turning into machines

Groups $\longrightarrow$ Machines $\longrightarrow$

# Groups turning into machines

Groups $\longrightarrow$ Machines $\longrightarrow$ Groups

# Miller machine

Let $G = \langle X \mid R \rangle$ be an f.p. group.

# Miller machine

Let $G = \langle X \mid R \rangle$ be an f.p. group.

Consider a new group $MG$.

# Miller machine

Let $G = \langle X \mid R \rangle$ be an f.p. group.

Consider a new group $MG$. The generators: tape letters $X$, state letter $q$, command letters $\theta_x, \theta_r, x \in X, r \in R$.

# Miller machine

Let $G = \langle X \mid R \rangle$ be an f.p. group.

Consider a new group $MG$. The generators: tape letters $X$, state letter $q$, command letters $\theta_x, \theta_r, x \in X, r \in R$.

The defining relations:

# Miller machine

Let $G = \langle X \mid R \rangle$ be an f.p. group.

Consider a new group $MG$. The generators: tape letters $X$, state letter $q$, command letters $\theta_x, \theta_r, x \in X, r \in R$.

The defining relations:

$$qx\theta_x = \theta_x xq, \qquad q\theta_r = \theta_r qr, \qquad x\theta = \theta x$$

# Miller machine

Let $G = \langle X \mid R \rangle$ be an f.p. group.

Consider a new group $MG$. The generators: tape letters $X$, state letter $q$, command letters $\theta_x, \theta_r, x \in X, r \in R$.

The defining relations:

$$qx\theta_x = \theta_x xq, \qquad q\theta_r = \theta_r qr, \qquad x\theta = \theta x$$

We can draw these relations as follows.

# Miller machine

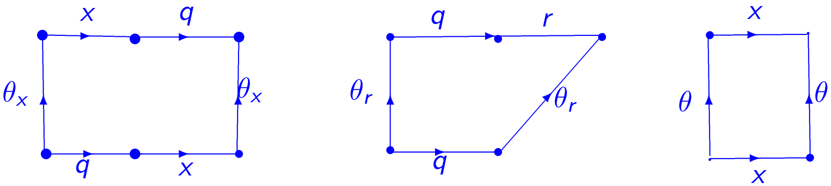
Let $G = \langle X \mid R \rangle$ be an f.p. group.

Consider a new group $MG$. The generators: tape letters $X$, state letter $q$, command letters $\theta_x, \theta_r, x \in X, r \in R$.

The defining relations:

$$q x \theta_x = \theta_x x q, \qquad q\theta_r = \theta_r q r, \qquad x\theta = \theta x$$

We can draw these relations as follows.



# Why is it a machine?

Let us show that this is a machine

# Why is it a machine?

Let us show that this is a machine checking whether a word is equal to 1 in $G$

# Why is it a machine?

Let us show that this is a machine checking whether a word is equal to 1 in $G$

Start with any word $uqv = y...xq...z$

# Why is it a machine?

Let us show that this is a machine checking whether a word is equal to 1 in $G$

Start with any word $uqv = y...xq...z$ where $uv = 1$ in $G$

# Why is it a machine?

Let us show that this is a machine checking whether a word is equal to 1 in $G$

Start with any word $uqv = y...xq...z$ where $uv = 1$ in $G$

$y$ $\ldots$ $xq$ $\ldots$ $z$

Figure: Deduction $uqv \to ... \to q$ if $uv = 1$ in $G$. Here $u = y...x$, $v = ...z$.

# Why is it a machine?

Let us show that this is a machine checking whether a word is equal to 1 in $G$

Start with any word $uqv = y ... xq ... z$ where $uv = 1$ in $G$

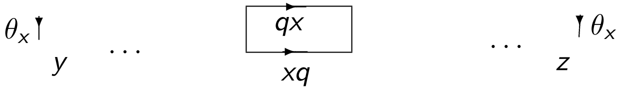



Figure: Deduction $uqv \to ... \to q$ if $uv = 1$ in $G$. Here $u = y ... x$, $v = ... z$.

# Why is it a machine?

Let us show that this is a machine checking whether a word is equal to 1 in $G$

Start with any word $uqv = y...xq...z$ where $uv = 1$ in $G$

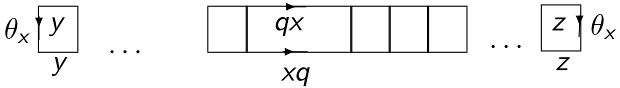



Figure: Deduction $uqv \to ... \to q$ if $uv = 1$ in $G$. Here $u = y...x$, $v = ...z$.

# Why is it a machine?

Let us show that this is a machine checking whether a word is equal to 1 in $G$

Start with any word $uqv = y...xq...z$ where $uv = 1$ in $G$

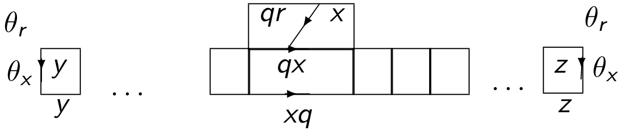



Figure: Deduction $uqv \to ... \to q$ if $uv = 1$ in $G$. Here $u = y...x$, $v = ...z$.

# Why is it a machine?

Let us show that this is a machine checking whether a word is equal to 1 in $G$

Start with any word $uqv = y...xq...z$ where $uv = 1$ in $G$

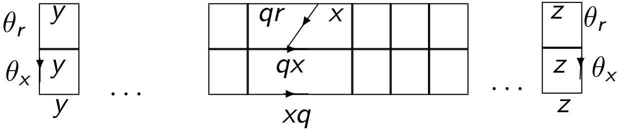



Figure: Deduction $uqv \to ... \to q$ if $uv = 1$ in $G$. Here $u = y...x$, $v = ...z$.

# Why is it a machine?

Let us show that this is a machine checking whether a word is equal to 1 in $G$

Start with any word $uqv = y...xq...z$ where $uv = 1$ in $G$

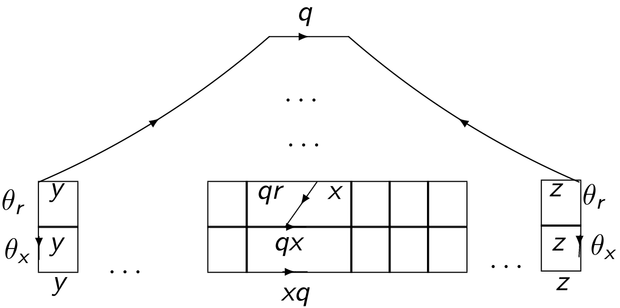



Figure: Deduction $uqv \to ... \to q$ if $uv = 1$ in $G$. Here $u = y...x$, $v = ...z$.

# Why is it a machine?

Let us show that this is a machine checking whether a word is equal to 1 in $G$

Start with any word $uqv = y...xq...z$ where $uv = 1$ in $G$

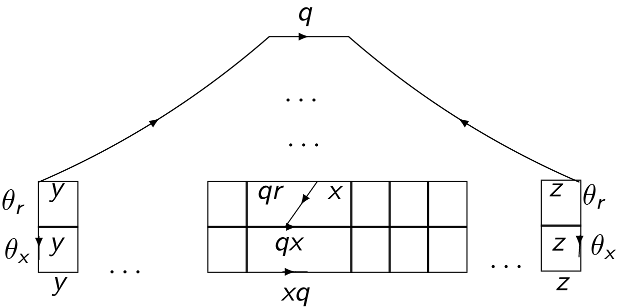



Figure: Deduction $uqv \to ... \to q$ if $uv = 1$ in $G$. Here $u = y...x$, $v = ...z$.

**Theorem.**(Miller)

# Why is it a machine?

Let us show that this is a machine checking whether a word is equal to 1 in $G$

Start with any word $uqv = y...xq...z$ where $uv = 1$ in $G$

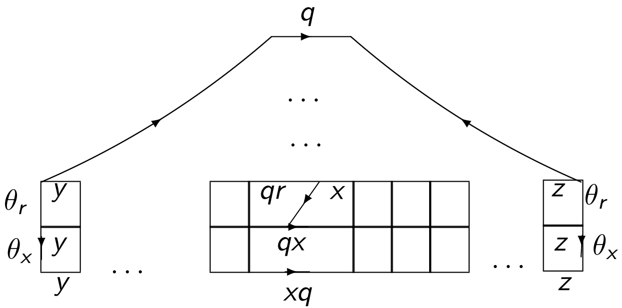



Figure: Deduction $uqv \to ... \to q$ if $uv = 1$ in $G$. Here $u = y...x$, $v = ...z$.

**Theorem.**(Miller) The group $MG$ has solvable conjugacy problem iff

# Why is it a machine?

Let us show that this is a machine checking whether a word is equal to 1 in $G$

Start with any word $uqv = y...xq...z$ where $uv = 1$ in $G$

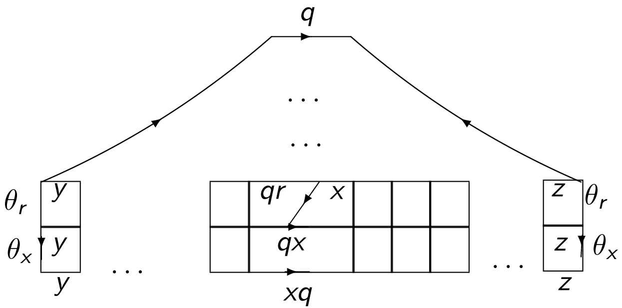



Figure: Deduction $uqv \to ... \to q$ if $uv = 1$ in $G$. Here $u = y...x$, $v = ...z$.

**Theorem.**(Miller) The group $MG$ has solvable conjugacy problem iff $G$ has solvable word problem.

# $S$-machines

Now we define machines that are groups.

# $S$-machines

Now we define machines that are groups. Here is a definition of a simplest (1-tape, no semigroup part) version of $S$-machines.

# $S$-machines

Now we define machines that are groups. Here is a definition of a simplest (1-tape, no semigroup part) version of $S$-machines.

**Definition.** An $S$-machine is an HNN extension of a free group.

# $S$-machines

Now we define machines that are groups. Here is a definition of a simplest (1-tape, no semigroup part) version of $S$-machines.

**Definition.** An $S$-machine is an HNN extension of a free group.
Generators: tape letters $X$,

# $S$-machines

Now we define machines that are groups. Here is a definition of a simplest (1-tape, no semigroup part) version of $S$-machines.

**Definition.** An $S$-machine is an HNN extension of a free group. Generators: tape letters $X$, state letters $Q$,

# $S$-machines

Now we define machines that are groups. Here is a definition of a simplest (1-tape, no semigroup part) version of $S$-machines.

**Definition.** An $S$-machine is an HNN extension of a free group. Generators: tape letters $X$, state letters $Q$, command letters $\Theta$.

# $S$-machines

Now we define machines that are groups. Here is a definition of a simplest (1-tape, no semigroup part) version of $S$-machines.

**Definition.** An $S$-machine is an HNN extension of a free group. Generators: tape letters $X$, state letters $Q$, command letters $\Theta$.

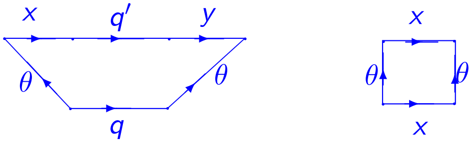
Relations are of one of the forms:

# $S$-machines

Now we define machines that are groups. Here is a definition of a simplest (1-tape, no semigroup part) version of $S$-machines.

**Definition.** An $S$-machine is an HNN extension of a free group. Generators: tape letters $X$, state letters $Q$, command letters $\Theta$.

Relations are of one of the forms:

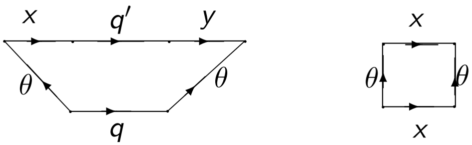


# $S$-machines

Now we define machines that are groups. Here is a definition of a simplest (1-tape, no semigroup part) version of $S$-machines.

**Definition.** An $S$-machine is an HNN extension of a free group. Generators: tape letters $X$, state letters $Q$, command letters $\Theta$.

Relations are of one of the forms:




**The main idea:** *$S$-machines are much easier to use as building blocks of groups than Turing machines.*

# Applications: Dehn functions of groups

**Definition. (Madlener-Otto, Gersten, Gromov)**

# Applications: Dehn functions of groups

**Definition. (Madlener-Otto, Gersten, Gromov)** Let $G = \langle X \mid R \rangle$ be a f.p. group,

# Applications: Dehn functions of groups

**Definition. (Madlener-Otto, Gersten, Gromov)** Let $G = \langle X \mid R \rangle$ be a f.p. group, $w$ be a word in $X$, $w = 1$ in $G$.

# Applications: Dehn functions of groups

**Definition. (Madlener-Otto, Gersten, Gromov)** Let $G = \langle X \mid R \rangle$ be a f.p. group, $w$ be a word in $X$, $w = 1$ in $G$. The *area*

# Applications: Dehn functions of groups

**Definition. (Madlener-Otto, Gersten, Gromov)** Let $G = \langle X \mid R \rangle$ be a f.p. group, $w$ be a word in $X$, $w = 1$ in $G$. The *area* of $w$ is the minimal number of cells in a van Kampen diagram with boundary label $w$.

# Applications: Dehn functions of groups

**Definition. (Madlener-Otto, Gersten, Gromov)** Let $G = \langle X \mid R \rangle$ be a f.p. group, $w$ be a word in $X$, $w = 1$ in $G$. The *area* of $w$ is the minimal number of cells in a van Kampen diagram with boundary label $w$. That is how long it takes to deduce the equality $w = 1$ from the defining relations.

# Applications: Dehn functions of groups

**Definition. (Madlener-Otto, Gersten, Gromov)** Let $G = \langle X \mid R \rangle$ be a f.p. group, $w$ be a word in $X$, $w = 1$ in $G$. The *area* of $w$ is the minimal number of cells in a van Kampen diagram with boundary label $w$. That is how long it takes to deduce the equality $w = 1$ from the defining relations.

**Definition. (Dehn function)**

# Applications: Dehn functions of groups

**Definition. (Madlener-Otto, Gersten, Gromov)** Let $G = \langle X \mid R \rangle$ be a f.p. group, $w$ be a word in $X$, $w = 1$ in $G$. The *area* of $w$ is the minimal number of cells in a van Kampen diagram with boundary label $w$. That is how long it takes to deduce the equality $w = 1$ from the defining relations.

**Definition. (Dehn function)** For any $n \geq 1$ let $d(n)$ be the largest area of a word $w$ of length at most $n$.

# Applications: Dehn functions of groups

**Definition. (Madlener-Otto, Gersten, Gromov)** Let $G = \langle X \mid R \rangle$ be a f.p. group, $w$ be a word in $X$, $w = 1$ in $G$. The *area* of $w$ is the minimal number of cells in a van Kampen diagram with boundary label $w$. That is how long it takes to deduce the equality $w = 1$ from the defining relations.

**Definition. (Dehn function)** For any $n \geq 1$ let $d(n)$ be the largest area of a word $w$ of length at most $n$.

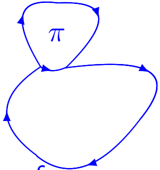
**Example. Surface group** The Dehn function is linear.

# Applications: Dehn functions of groups

**Definition. (Madlener-Otto, Gersten, Gromov)** Let $G = \langle X \mid R \rangle$ be a f.p. group, $w$ be a word in $X$, $w = 1$ in $G$. The *area* of $w$ is the minimal number of cells in a van Kampen diagram with boundary label $w$. That is how long it takes to deduce the equality $w = 1$ from the defining relations.

**Definition. (Dehn function)** For any $n \geq 1$ let $d(n)$ be the largest area of a word $w$ of length at most $n$.

**Example. Surface group** The Dehn function is linear.



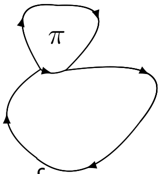

A typical diagram over the surface group presentation (genus $> 1$)

# Applications: Dehn functions of groups

**Definition. (Madlener-Otto, Gersten, Gromov)** Let $G = \langle X \mid R \rangle$ be a f.p. group, $w$ be a word in $X$, $w = 1$ in $G$. The *area* of $w$ is the minimal number of cells in a van Kampen diagram with boundary label $w$. That is how long it takes to deduce the equality $w = 1$ from the defining relations.

**Definition. (Dehn function)** For any $n \geq 1$ let $d(n)$ be the largest area of a word $w$ of length at most $n$.

**Example. Surface group** The Dehn function is linear.




A typical diagram over the surface group presentation (genus $> 1$)

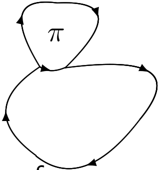
**Definition.**

# Applications: Dehn functions of groups

**Definition. (Madlener-Otto, Gersten, Gromov)** Let $G = \langle X \mid R \rangle$ be a f.p. group, $w$ be a word in $X$, $w = 1$ in $G$. The *area* of $w$ is the minimal number of cells in a van Kampen diagram with boundary label $w$. That is how long it takes to deduce the equality $w = 1$ from the defining relations.

**Definition. (Dehn function)** For any $n \geq 1$ let $d(n)$ be the largest area of a word $w$ of length at most $n$.

**Example. Surface group** The Dehn function is linear.




A typical diagram over the surface group presentation (genus $> 1$)

**Definition.** A group is *hyperbolic* if its Dehn function is linear.

# Dehn functions

**Example.**

# Dehn functions

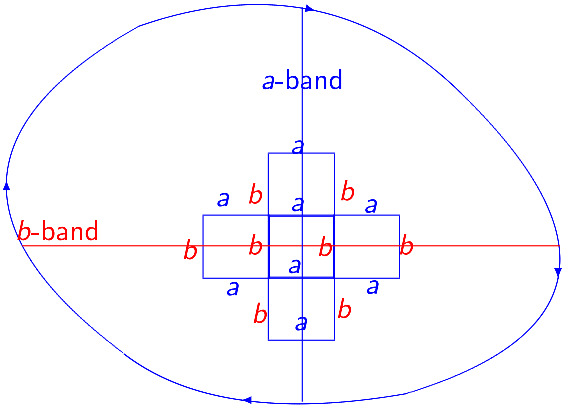
**Example.** The Dehn function of the Abelian group $\langle a, b \mid ab = ba \rangle$

# Dehn functions

**Example.** The Dehn function of the Abelian group $\langle a, b \mid ab = ba \rangle$ is quadratic:

# Dehn functions

**Example.** The Dehn function of the Abelian group $\langle a, b \mid ab = ba \rangle$ is quadratic:



# Dehn functions of $S$-machines

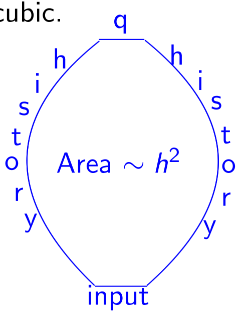
**Observation. (Rips)**

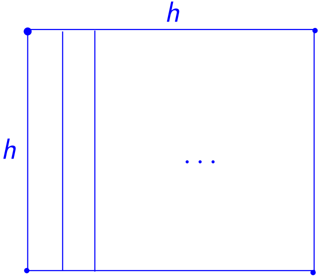
# Dehn functions of $S$-machines

**Observation. (Rips)** The Dehn function of any $S$-machine is at most cubic.

# Dehn functions of $S$-machines

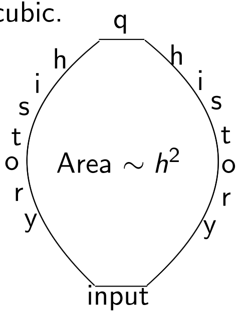
**Observation. (Rips)** The Dehn function of any $S$-machine is at most cubic.

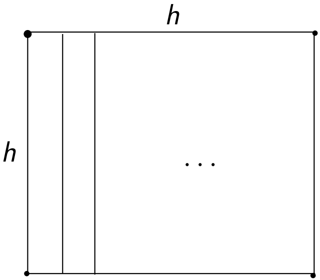


# Dehn functions of $S$-machines

**Observation. (Rips)** The Dehn function of any $S$-machine is at most cubic.

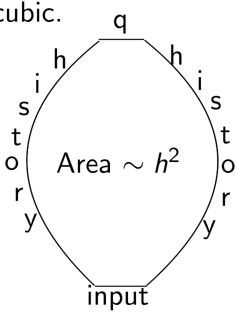


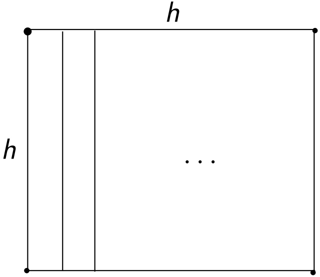




**Theorem. (Olshanskii, S.)**

# Dehn functions of $S$-machines

**Observation. (Rips)** The Dehn function of any $S$-machine is at most cubic.

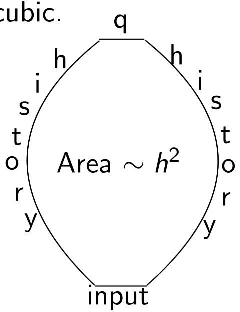


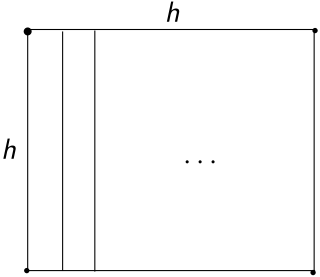




**Theorem. (Olshanskii, S.)** There exists a f.p. group with Dehn function $n^2 \log n$

# Dehn functions of $S$-machines

**Observation. (Rips)** The Dehn function of any $S$-machine is at most cubic.

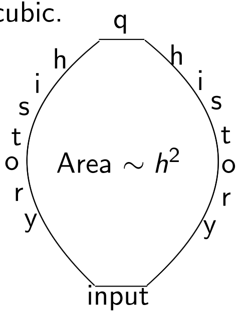


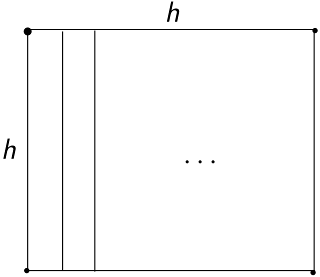




**Theorem. (Olshanskii, S.)** There exists a f.p. group with Dehn function $n^2 \log n$ and undecidable conjugacy problem.

# Dehn functions of $S$-machines

**Observation. (Rips)** The Dehn function of any $S$-machine is at most cubic.



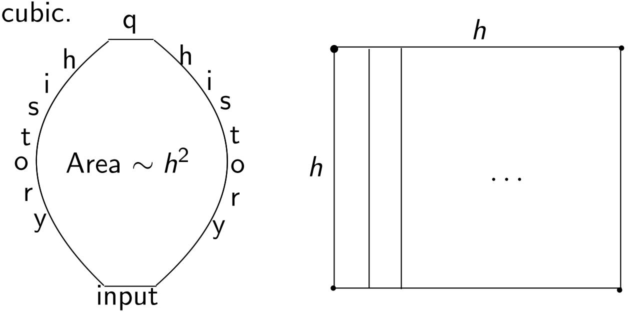




**Theorem. (Olshanskii, S.)** There exists a f.p. group with Dehn function $n^2 \log n$ and undecidable conjugacy problem.

We use some (Vassiliev-type) invariants of chord diagrams

# Dehn functions of $S$-machines

**Observation. (Rips)** The Dehn function of any $S$-machine is at most cubic.




**Theorem. (Olshanskii, S.)** There exists a f.p. group with Dehn function $n^2 \log n$ and undecidable conjugacy problem.

We use some (Vassiliev-type) invariants of chord diagrams to obtain the upper bound.

# Corollaries about conjugacy problem

**Conjecture.**

# Corollaries about conjugacy problem

**Conjecture.** Every group with Dehn function $<_e n^2 \log n$

# Corollaries about conjugacy problem

**Conjecture.** Every group with Dehn function $<_e n^2 \log n$ has solvable conjugacy problem.

# Corollaries about conjugacy problem

**Conjecture.** Every group with Dehn function $<_e n^2 \log n$ has solvable conjugacy problem.

O+S: proved for HNN extensions of free groups.

# Corollaries about conjugacy problem

**Conjecture.** Every group with Dehn function $<_e n^2 \log n$ has solvable conjugacy problem.

O+S: proved for HNN extensions of free groups.

Together with a result of Bridson+Groves

# Corollaries about conjugacy problem

**Conjecture.** Every group with Dehn function $<_e n^2 \log n$ has solvable conjugacy problem.

O+S: proved for HNN extensions of free groups.

Together with a result of Bridson+Groves gives a proof of solvability of conjugacy problem

# Corollaries about conjugacy problem

**Conjecture.** Every group with Dehn function $<_e n^2 \log n$ has solvable conjugacy problem.

O+S: proved for HNN extensions of free groups.

Together with a result of Bridson+Groves gives a proof of solvability of conjugacy problem for cyclic extensions of free groups.

# Corollaries about conjugacy problem

**Conjecture.** Every group with Dehn function $<_e n^2 \log n$ has solvable conjugacy problem.

O+S: proved for HNN extensions of free groups.

Together with a result of Bridson+Groves gives a proof of solvability of conjugacy problem for cyclic extensions of free groups.

An earlier proof: Bogopolski, Martino, Maslakova, Ventura.

# Bigger Dehn functions

**Theorem. (Birget-S.)**

# Bigger Dehn functions

**Theorem. (Birget-S.)** Every Dehn function of a f.p. group

# Bigger Dehn functions

**Theorem. (Birget-S.)** Every Dehn function of a f.p. group is equivalent to the time function of a non-deterministic Turing machine.

# Bigger Dehn functions

**Theorem. (Birget-S.)** Every Dehn function of a f.p. group is equivalent to the time function of a non-deterministic Turing machine.

**Theorem. (S., Birget, Rips, Ann. of Math., 2002)**

# Bigger Dehn functions

**Theorem. (Birget-S.)** Every Dehn function of a f.p. group is equivalent to the time function of a non-deterministic Turing machine.

**Theorem. (S., Birget, Rips, Ann. of Math., 2002)** If a time function is superadditive and $\succ n^4$

# Bigger Dehn functions

**Theorem. (Birget-S.)** Every Dehn function of a f.p. group is equivalent to the time function of a non-deterministic Turing machine.

**Theorem. (S., Birget, Rips, Ann. of Math., 2002)** If a time function is superadditive and $\succ n^4$ then it is equivalent to the Dehn function of a f.p. group.

# Bigger Dehn functions

**Theorem. (Birget-S.)** Every Dehn function of a f.p. group is equivalent to the time function of a non-deterministic Turing machine.

**Theorem. (S., Birget, Rips, Ann. of Math., 2002)** If a time function is superadditive and $\succ n^4$ then it is equivalent to the Dehn function of a f.p. group.

**Theorem. (S.)**

# Bigger Dehn functions

**Theorem. (Birget-S.)** Every Dehn function of a f.p. group is equivalent to the time function of a non-deterministic Turing machine.

**Theorem. (S., Birget, Rips, Ann. of Math., 2002)** If a time function is superadditive and $\succ n^4$ then it is equivalent to the Dehn function of a f.p. group.

**Theorem. (S.)** Every Turing machine is polynomially equivalent to an $S$-machine

# Bigger Dehn functions

**Theorem. (Birget-S.)** Every Dehn function of a f.p. group is equivalent to the time function of a non-deterministic Turing machine.

**Theorem. (S., Birget, Rips, Ann. of Math., 2002)** If a time function is superadditive and $\succ n^4$ then it is equivalent to the Dehn function of a f.p. group.

**Theorem. (S.)** Every Turing machine is polynomially equivalent to an $S$-machine with one tape and one state

# Bigger Dehn functions

**Theorem. (Birget-S.)** Every Dehn function of a f.p. group is equivalent to the time function of a non-deterministic Turing machine.

**Theorem. (S., Birget, Rips, Ann. of Math., 2002)** If a time function is superadditive and $\succ n^4$ then it is equivalent to the Dehn function of a f.p. group.

**Theorem. (S.)** Every Turing machine is polynomially equivalent to an $S$-machine with one tape and one state (Miller machine).

# Bigger Dehn functions

**Theorem. (Birget-S.)** Every Dehn function of a f.p. group is equivalent to the time function of a non-deterministic Turing machine.

**Theorem. (S., Birget, Rips, Ann. of Math., 2002)** If a time function is superadditive and $\succ n^4$ then it is equivalent to the Dehn function of a f.p. group.

**Theorem. (S.)** Every Turing machine is polynomially equivalent to an $S$-machine with one tape and one state (Miller machine).

**Corollary. (S.)**

# Bigger Dehn functions

**Theorem. (Birget-S.)** Every Dehn function of a f.p. group is equivalent to the time function of a non-deterministic Turing machine.

**Theorem. (S., Birget, Rips, Ann. of Math., 2002)** If a time function is superadditive and $\succ n^4$ then it is equivalent to the Dehn function of a f.p. group.

**Theorem. (S.)** Every Turing machine is polynomially equivalent to an $S$-machine with one tape and one state (Miller machine).

**Corollary. (S.)** For every number $\alpha \geq 4$ that is computable in time $\leq 2^{2^n}$,

# Bigger Dehn functions

**Theorem. (Birget-S.)** Every Dehn function of a f.p. group is equivalent to the time function of a non-deterministic Turing machine.

**Theorem. (S., Birget, Rips, Ann. of Math., 2002)** If a time function is superadditive and $\succ n^4$ then it is equivalent to the Dehn function of a f.p. group.

**Theorem. (S.)** Every Turing machine is polynomially equivalent to an $S$-machine with one tape and one state (Miller machine).

**Corollary. (S.)** For every number $\alpha \geq 4$ that is computable in time $\leq 2^{2^n}$, there exists a f.p. group with Dehn function $n^\alpha$.

# Bigger Dehn functions

**Theorem. (Birget-S.)** Every Dehn function of a f.p. group is equivalent to the time function of a non-deterministic Turing machine.

**Theorem. (S., Birget, Rips, Ann. of Math., 2002)** If a time function is superadditive and $\succ n^4$ then it is equivalent to the Dehn function of a f.p. group.

**Theorem. (S.)** Every Turing machine is polynomially equivalent to an $S$-machine with one tape and one state (Miller machine).

**Corollary. (S.)** For every number $\alpha \geq 4$ that is computable in time $\leq 2^{2^n}$, there exists a f.p. group with Dehn function $n^\alpha$.
Conversely, every number in the isoperimetric spectrum

# Bigger Dehn functions

**Theorem. (Birget-S.)** Every Dehn function of a f.p. group is equivalent to the time function of a non-deterministic Turing machine.

**Theorem. (S., Birget, Rips, Ann. of Math., 2002)** If a time function is superadditive and $\succ n^4$ then it is equivalent to the Dehn function of a f.p. group.

**Theorem. (S.)** Every Turing machine is polynomially equivalent to an $S$-machine with one tape and one state (Miller machine).

**Corollary. (S.)** For every number $\alpha \geq 4$ that is computable in time $\leq 2^{2^n}$, there exists a f.p. group with Dehn function $n^\alpha$. Conversely, every number in the isoperimetric spectrum is computable in time $\leq 2^{2^{2^n}}$.

# Asymptotic cones

**Definition. (Gromov, van den Dries - Wilkie)**

# Asymptotic cones

**Definition. (Gromov, van den Dries - Wilkie)** An asymptotic cone of a group $G$ corresponding to a sequences of scalars $d_n \to \infty$

# Asymptotic cones

**Definition. (Gromov, van den Dries - Wilkie)** An asymptotic cone of a group $G$ corresponding to a sequences of scalars $d_n \to \infty$ is a Gromov-Hausdorff limit of the spaces $G/d_n$.

# Asymptotic cones

**Definition. (Gromov, van den Dries - Wilkie)** An asymptotic cone of a group $G$ corresponding to a sequences of scalars $d_n \to \infty$ is a Gromov-Hausdorff limit of the spaces $G/d_n$.

**Theorem. (Gromov)**

# Asymptotic cones

**Definition. (Gromov, van den Dries - Wilkie)** An asymptotic cone of a group $G$ corresponding to a sequences of scalars $d_n \to \infty$ is a Gromov-Hausdorff limit of the spaces $G/d_n$.

**Theorem. (Gromov)** If all a. c. of a group are simply connected

# Asymptotic cones

**Definition. (Gromov, van den Dries - Wilkie)** An asymptotic cone of a group $G$ corresponding to a sequences of scalars $d_n \to \infty$ is a Gromov-Hausdorff limit of the spaces $G/d_n$.

**Theorem. (Gromov)** If all a. c. of a group are simply connected then the group has polynomial Dehn function and linear isodiametric function.

# Asymptotic cones

**Definition. (Gromov, van den Dries - Wilkie)** An asymptotic cone of a group $G$ corresponding to a sequences of scalars $d_n \to \infty$ is a Gromov-Hausdorff limit of the spaces $G/d_n$.

**Theorem. (Gromov)** If all a. c. of a group are simply connected then the group has polynomial Dehn function and linear isodiametric function.

**Papasoglu:** The converse statement is true if the Dehn function is quadratic.

# Asymptotic cones

**Definition. (Gromov, van den Dries - Wilkie)** An asymptotic cone of a group $G$ corresponding to a sequences of scalars $d_n \to \infty$ is a Gromov-Hausdorff limit of the spaces $G/d_n$.

**Theorem. (Gromov)** If all a. c. of a group are simply connected then the group has polynomial Dehn function and linear isodiametric function.

**Papasoglu:** The converse statement is true if the Dehn function is quadratic.

**Theorem. (O+S, solving a problem of Druțu)**

# Asymptotic cones

**Definition. (Gromov, van den Dries - Wilkie)** An asymptotic cone of a group $G$ corresponding to a sequences of scalars $d_n \to \infty$ is a Gromov-Hausdorff limit of the spaces $G/d_n$.

**Theorem. (Gromov)** If all a. c. of a group are simply connected then the group has polynomial Dehn function and linear isodiametric function.

**Papasoglu:** The converse statement is true if the Dehn function is quadratic.

**Theorem. (O+S, solving a problem of Druțu)** There are f.p. groups with non-simply connected asymptotic cones

# Asymptotic cones

**Definition. (Gromov, van den Dries - Wilkie)** An asymptotic cone of a group $G$ corresponding to a sequences of scalars $d_n \to \infty$ is a Gromov-Hausdorff limit of the spaces $G/d_n$.

**Theorem. (Gromov)** If all a. c. of a group are simply connected then the group has polynomial Dehn function and linear isodiametric function.

**Papasoglu:** The converse statement is true if the Dehn function is quadratic.

**Theorem. (O+S, solving a problem of Druțu)** There are f.p. groups with non-simply connected asymptotic cones but Dehn functions arbitrary close to $n^2$

# Asymptotic cones

**Definition. (Gromov, van den Dries - Wilkie)** An asymptotic cone of a group $G$ corresponding to a sequences of scalars $d_n \to \infty$ is a Gromov-Hausdorff limit of the spaces $G/d_n$.

**Theorem. (Gromov)** If all a. c. of a group are simply connected then the group has polynomial Dehn function and linear isodiametric function.

**Papasoglu:** The converse statement is true if the Dehn function is quadratic.

**Theorem. (O+S, solving a problem of Druțu)** There are f.p. groups with non-simply connected asymptotic cones but Dehn functions arbitrary close to $n^2$ and linear isodiametric function.

# Asymptotic cones

**Definition. (Gromov, van den Dries - Wilkie)** An asymptotic cone of a group $G$ corresponding to a sequences of scalars $d_n \to \infty$ is a Gromov-Hausdorff limit of the spaces $G/d_n$.

**Theorem. (Gromov)** If all a. c. of a group are simply connected then the group has polynomial Dehn function and linear isodiametric function.

**Papasoglu:** The converse statement is true if the Dehn function is quadratic.

**Theorem. (O+S, solving a problem of Druțu)** There are f.p. groups with non-simply connected asymptotic cones but Dehn functions arbitrary close to $n^2$ and linear isodiametric function.

**Theorem (O+S)**

# Asymptotic cones

**Definition. (Gromov, van den Dries - Wilkie)** An asymptotic cone of a group $G$ corresponding to a sequences of scalars $d_n \to \infty$ is a Gromov-Hausdorff limit of the spaces $G/d_n$.

**Theorem. (Gromov)** If all a. c. of a group are simply connected then the group has polynomial Dehn function and linear isodiametric function.

**Papasoglu:** The converse statement is true if the Dehn function is quadratic.

**Theorem. (O+S, solving a problem of Druțu)** There are f.p. groups with non-simply connected asymptotic cones but Dehn functions arbitrary close to $n^2$ and linear isodiametric function.

**Theorem (O+S)** There are f.p. groups with undecidable word problem

# Asymptotic cones

**Definition. (Gromov, van den Dries - Wilkie)** An asymptotic cone of a group $G$ corresponding to a sequences of scalars $d_n \to \infty$ is a Gromov-Hausdorff limit of the spaces $G/d_n$.

**Theorem. (Gromov)** If all a. c. of a group are simply connected then the group has polynomial Dehn function and linear isodiametric function.

**Papasoglu:** The converse statement is true if the Dehn function is quadratic.

**Theorem. (O+S, solving a problem of Druțu)** There are f.p. groups with non-simply connected asymptotic cones but Dehn functions arbitrary close to $n^2$ and linear isodiametric function.

**Theorem (O+S)** There are f.p. groups with undecidable word problem but Dehn function $f(n)$ bounded by $Cn^2$ for infinitely many $n$'s.

# Asymptotic cones

**Definition. (Gromov, van den Dries - Wilkie)** An asymptotic cone of a group $G$ corresponding to a sequences of scalars $d_n \to \infty$ is a Gromov-Hausdorff limit of the spaces $G/d_n$.

**Theorem. (Gromov)** If all a. c. of a group are simply connected then the group has polynomial Dehn function and linear isodiametric function.

**Papasoglu:** The converse statement is true if the Dehn function is quadratic.

**Theorem. (O+S, solving a problem of Druțu)** There are f.p. groups with non-simply connected asymptotic cones but Dehn functions arbitrary close to $n^2$ and linear isodiametric function.

**Theorem (O+S)** There are f.p. groups with undecidable word problem but Dehn function $f(n)$ bounded by $Cn^2$ for infinitely many $n$'s. Such a group has at least two non-homeomorphic a. c.:

# Asymptotic cones

**Definition. (Gromov, van den Dries - Wilkie)** An asymptotic cone of a group $G$ corresponding to a sequences of scalars $d_n \to \infty$ is a Gromov-Hausdorff limit of the spaces $G/d_n$.

**Theorem. (Gromov)** If all a. c. of a group are simply connected then the group has polynomial Dehn function and linear isodiametric function.

**Papasoglu:** The converse statement is true if the Dehn function is quadratic.

**Theorem. (O+S, solving a problem of Druțu)** There are f.p. groups with non-simply connected asymptotic cones but Dehn functions arbitrary close to $n^2$ and linear isodiametric function.

**Theorem (O+S)** There are f.p. groups with undecidable word problem but Dehn function $f(n)$ bounded by $Cn^2$ for infinitely many $n$'s. Such a group has at least two non-homeomorphic a. c.: one simply connected and one non-simply connected.

# Asymptotic cones

**Definition. (Gromov, van den Dries - Wilkie)** An asymptotic cone of a group $G$ corresponding to a sequences of scalars $d_n \to \infty$ is a Gromov-Hausdorff limit of the spaces $G/d_n$.

**Theorem. (Gromov)** If all a. c. of a group are simply connected then the group has polynomial Dehn function and linear isodiametric function.

**Papasoglu:** The converse statement is true if the Dehn function is quadratic.

**Theorem. (O+S, solving a problem of Druțu)** There are f.p. groups with non-simply connected asymptotic cones but Dehn functions arbitrary close to $n^2$ and linear isodiametric function.

**Theorem (O+S)** There are f.p. groups with undecidable word problem but Dehn function $f(n)$ bounded by $Cn^2$ for infinitely many $n$'s. Such a group has at least two non-homeomorphic a. c.: one simply connected and one non-simply connected.

**Kramer, Shelah, Tent, Thomas:**

# Asymptotic cones

**Definition. (Gromov, van den Dries - Wilkie)** An asymptotic cone of a group $G$ corresponding to a sequences of scalars $d_n \to \infty$ is a Gromov-Hausdorff limit of the spaces $G/d_n$.

**Theorem. (Gromov)** If all a. c. of a group are simply connected then the group has polynomial Dehn function and linear isodiametric function.

**Papasoglu:** The converse statement is true if the Dehn function is quadratic.

**Theorem. (O+S, solving a problem of Druțu)** There are f.p. groups with non-simply connected asymptotic cones but Dehn functions arbitrary close to $n^2$ and linear isodiametric function.

**Theorem (O+S)** There are f.p. groups with undecidable word problem but Dehn function $f(n)$ bounded by $Cn^2$ for infinitely many $n$'s. Such a group has at least two non-homeomorphic a. c.: one simply connected and one non-simply connected.

**Kramer, Shelah, Tent, Thomas:** assuming CH is *not* true.

# Embeddings

**Theorem. (Birget, Rips, Olshanskii, S., Ann. of Math., 2002)**

# Embeddings

**Theorem. (Birget, Rips, Olshanskii, S., Ann. of Math., 2002)** A finitely generated group has word problem in NP

# Embeddings

**Theorem. (Birget, Rips, Olshanskii, S., Ann. of Math., 2002)** A finitely generated group has word problem in NP iff it is inside a finitely presented group with polynomial Dehn function.

# Embeddings

**Theorem. (Birget, Rips, Olshanskii, S., Ann. of Math., 2002)** A finitely generated group has word problem in NP iff it is inside a finitely presented group with polynomial Dehn function.

**Corollary.**

# Embeddings

**Theorem. (Birget, Rips, Olshanskii, S., Ann. of Math., 2002)** A finitely generated group has word problem in NP iff it is inside a finitely presented group with polynomial Dehn function.

**Corollary.** There exists an NP-complete f.p. group.

# Embeddings

**Theorem. (Birget, Rips, Olshanskii, S., Ann. of Math., 2002)** A finitely generated group has word problem in NP iff it is inside a finitely presented group with polynomial Dehn function.

**Corollary.** There exists an NP-complete f.p. group.

**Corollary.**

# Embeddings

**Theorem. (Birget, Rips, Olshanskii, S., Ann. of Math., 2002)** A finitely generated group has word problem in NP iff it is inside a finitely presented group with polynomial Dehn function.

**Corollary.** There exists an NP-complete f.p. group.

**Corollary.** If a word problem in a f.g. group can be solved in NP-time by a smart algorithm,

# Embeddings

**Theorem. (Birget, Rips, Olshanskii, S., Ann. of Math., 2002)** A finitely generated group has word problem in NP iff it is inside a finitely presented group with polynomial Dehn function.

**Corollary.** There exists an NP-complete f.p. group.

**Corollary.** If a word problem in a f.g. group can be solved in NP-time by a smart algorithm, it can be solved in NP-time by the obvious algorithm

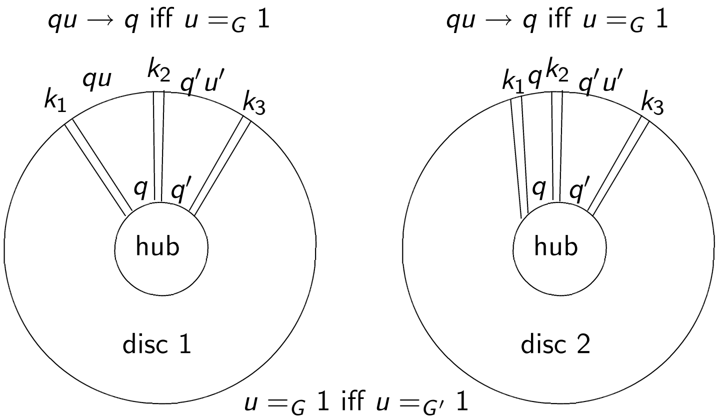
# Embeddings

**Theorem. (Birget, Rips, Olshanskii, S., Ann. of Math., 2002)** A finitely generated group has word problem in NP iff it is inside a finitely presented group with polynomial Dehn function.

**Corollary.** There exists an NP-complete f.p. group.

**Corollary.** If a word problem in a f.g. group can be solved in NP-time by a smart algorithm, it can be solved in NP-time by the obvious algorithm involving relations of a bigger group.

# Idea of the proof



# Collins' problem

The conjugacy problem is much harder to preserve under embeddings.

# Collins' problem

The conjugacy problem is much harder to preserve under embeddings.

Collins-Miller and Gorjaga-Kirkinskiĭ: even subgroups of index 2

# Collins' problem

The conjugacy problem is much harder to preserve under embeddings.

Collins-Miller and Gorjaga-Kirkinskiĭ: even subgroups of index 2 of finitely presented groups do not inherit solvability or unsolvability of the conjugacy problem.

# Collins' problem

The conjugacy problem is much harder to preserve under embeddings.

Collins-Miller and Gorjaga-Kirkinskiĭ: even subgroups of index 2 of finitely presented groups do not inherit solvability or unsolvability of the conjugacy problem.

D. Collins (1976)

# Collins' problem

The conjugacy problem is much harder to preserve under embeddings.

Collins-Miller and Gorjaga-Kirkinskiĭ: even subgroups of index 2 of finitely presented groups do not inherit solvability or unsolvability of the conjugacy problem.

D. Collins (1976) *Does there exist a version of the Higman embedding theorem in which the degree of unsolvability of the conjugacy problem is preserved?*

# Collins' problem

The conjugacy problem is much harder to preserve under embeddings.

Collins-Miller and Gorjaga-Kirkinskiĭ: even subgroups of index 2 of finitely presented groups do not inherit solvability or unsolvability of the conjugacy problem.

D. Collins (1976) *Does there exist a version of the Higman embedding theorem in which the degree of unsolvability of the conjugacy problem is preserved?*

**Theorem (O+S, Memoirs of AMS, 2004)**

# Collins' problem

The conjugacy problem is much harder to preserve under embeddings.

Collins-Miller and Gorjaga-Kirkinskiĭ: even subgroups of index 2 of finitely presented groups do not inherit solvability or unsolvability of the conjugacy problem.

D. Collins (1976) *Does there exist a version of the Higman embedding theorem in which the degree of unsolvability of the conjugacy problem is preserved?*

**Theorem (O+S, Memoirs of AMS, 2004)** A finitely generated group $H$ has solvable conjugacy problem

# Collins' problem

The conjugacy problem is much harder to preserve under embeddings.

Collins-Miller and Gorjaga-Kirkinskiĭ: even subgroups of index 2 of finitely presented groups do not inherit solvability or unsolvability of the conjugacy problem.

D. Collins (1976) *Does there exist a version of the Higman embedding theorem in which the degree of unsolvability of the conjugacy problem is preserved?*

**Theorem (O+S, Memoirs of AMS, 2004)** A finitely generated group $H$ has solvable conjugacy problem if and only if it is Frattini embedded into a finitely presented group $G$ with solvable conjugacy problem.

# Collins problem: the construction

- Embed $H$ into a finitely presented group $H_1$.

# Collins problem: the construction

- Embed $H$ into a finitely presented group $H_1$.
- Use the Miller $S$-machine $M(H_1)$ to solve the word problem in $H$.

# Collins problem: the construction

- Embed $H$ into a finitely presented group $H_1$.
- Use the Miller $S$-machine $M(H_1)$ to solve the word problem in $H$.
- Use Boone-Novikov to make a part of $M(H_1)$ act as TM.

# Collins problem: the construction

- Embed $H$ into a finitely presented group $H_1$.
- Use the Miller $S$-machine $M(H_1)$ to solve the word problem in $H$.
- Use Boone-Novikov to make a part of $M(H_1)$ act as TM.
- Embed $H$ into a f.p. group $G$ using the new machine.

# Collins problem: the construction

- Embed $H$ into a finitely presented group $H_1$.
- Use the Miller $S$-machine $M(H_1)$ to solve the word problem in $H$.
- Use Boone-Novikov to make a part of $M(H_1)$ act as TM.
- Embed $H$ into a f.p. group $G$ using the new machine.
- Use Makanin-Razborov to analyze conjugacy problem for trapezia.

# Collins problem: the construction

- Embed $H$ into a finitely presented group $H_1$.
- Use the Miller $S$-machine $M(H_1)$ to solve the word problem in $H$.
- Use Boone-Novikov to make a part of $M(H_1)$ act as TM.
- Embed $H$ into a f.p. group $G$ using the new machine.
- Use Makanin-Razborov to analyze conjugacy problem for trapezia.
- Analyze annular diagrams to solve conjugacy problem.

# Collins problem: the construction

- Embed $H$ into a finitely presented group $H_1$.
- Use the Miller $S$-machine $M(H_1)$ to solve the word problem in $H$.
- Use Boone-Novikov to make a part of $M(H_1)$ act as TM.
- Embed $H$ into a f.p. group $G$ using the new machine.
- Use Makanin-Razborov to analyze conjugacy problem for trapezia.
- Analyze annular diagrams to solve conjugacy problem.

**Problem.**

# Collins problem: the construction

- Embed $H$ into a finitely presented group $H_1$.
- Use the Miller $S$-machine $M(H_1)$ to solve the word problem in $H$.
- Use Boone-Novikov to make a part of $M(H_1)$ act as TM.
- Embed $H$ into a f.p. group $G$ using the new machine.
- Use Makanin-Razborov to analyze conjugacy problem for trapezia.
- Analyze annular diagrams to solve conjugacy problem.

**Problem.** Is there a version of Higman embedding preserving the complexity of conjugacy problem?

# von Neumann's problem: history

**Hausdorff, Banach, Tarski:**

# von Neumann's problem: history

**Hausdorff, Banach, Tarski:** One can cut a ball into several pieces and

# von Neumann's problem: history

**Hausdorff, Banach, Tarski:** One can cut a ball into several pieces and move them to assemble two balls of the same size.

# von Neumann's problem: history

**Hausdorff, Banach, Tarski:** One can cut a ball into several pieces and move them to assemble two balls of the same size.

**von Neumann:**

# von Neumann's problem: history

**Hausdorff, Banach, Tarski:** One can cut a ball into several pieces and move them to assemble two balls of the same size.

**von Neumann:** The reason is that the group of isometries of the ball

# von Neumann's problem: history

**Hausdorff, Banach, Tarski:** One can cut a ball into several pieces and move them to assemble two balls of the same size.

**von Neumann:** The reason is that the group of isometries of the ball is not *amenable*,

# von Neumann's problem: history

**Hausdorff, Banach, Tarski:** One can cut a ball into several pieces and move them to assemble two balls of the same size.

**von Neumann:** The reason is that the group of isometries of the ball is not *amenable*, i.e. there is no finitely additive left invariant probability measure on the set of all subsets of the group.

# von Neumann's problem: history

**Hausdorff, Banach, Tarski:** One can cut a ball into several pieces and move them to assemble two balls of the same size.
**von Neumann:** The reason is that the group of isometries of the ball is not *amenable*, i.e. there is no finitely additive left invariant probability measure on the set of all subsets of the group.

The reason for non-amenability:

# von Neumann's problem: history

**Hausdorff, Banach, Tarski:** One can cut a ball into several pieces and move them to assemble two balls of the same size.
**von Neumann:** The reason is that the group of isometries of the ball is not *amenable*, i.e. there is no finitely additive left invariant probability measure on the set of all subsets of the group.

The reason for non-amenability: existence of non-cyclic free subgroups.

# von Neumann's problem: history

**Hausdorff, Banach, Tarski:** One can cut a ball into several pieces and move them to assemble two balls of the same size.
**von Neumann:** The reason is that the group of isometries of the ball is not *amenable*, i.e. there is no finitely additive left invariant probability measure on the set of all subsets of the group.

The reason for non-amenability: existence of non-cyclic free subgroups.

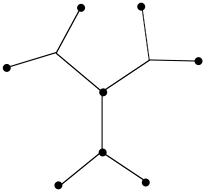
**Another definition. (Gromov):**

# von Neumann's problem: history

**Hausdorff, Banach, Tarski:** One can cut a ball into several pieces and move them to assemble two balls of the same size.
**von Neumann:** The reason is that the group of isometries of the ball is not *amenable*, i.e. there is no finitely additive left invariant probability measure on the set of all subsets of the group.

The reason for non-amenability: existence of non-cyclic free subgroups.

**Another definition. (Gromov):**

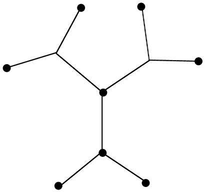


# von Neumann's problem: history

**Hausdorff, Banach, Tarski:** One can cut a ball into several pieces and move them to assemble two balls of the same size.
**von Neumann:** The reason is that the group of isometries of the ball is not *amenable*, i.e. there is no finitely additive left invariant probability measure on the set of all subsets of the group.

The reason for non-amenability: existence of non-cyclic free subgroups.

**Another definition. (Gromov):**



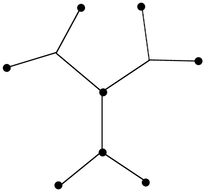
**Question. (Beatles)**

# von Neumann's problem: history

**Hausdorff, Banach, Tarski:** One can cut a ball into several pieces and move them to assemble two balls of the same size.
**von Neumann:** The reason is that the group of isometries of the ball is not *amenable*, i.e. there is no finitely additive left invariant probability measure on the set of all subsets of the group.

The reason for non-amenability: existence of non-cyclic free subgroups.

**Another definition. (Gromov):**



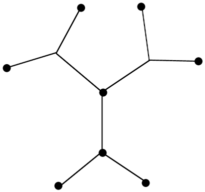
**Question. (Beatles)** Why can't we do it in the road?

# von Neumann's problem: history

**Hausdorff, Banach, Tarski:** One can cut a ball into several pieces and move them to assemble two balls of the same size.
**von Neumann:** The reason is that the group of isometries of the ball is not *amenable*, i.e. there is no finitely additive left invariant probability measure on the set of all subsets of the group.

The reason for non-amenability: existence of non-cyclic free subgroups.

**Another definition. (Gromov):**



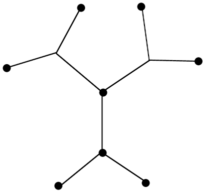
**Question. (Beatles)** Why can't we do it in the road?
**Answer:**

# von Neumann's problem: history

**Hausdorff, Banach, Tarski:** One can cut a ball into several pieces and move them to assemble two balls of the same size.
**von Neumann:** The reason is that the group of isometries of the ball is not *amenable*, i.e. there is no finitely additive left invariant probability measure on the set of all subsets of the group.

The reason for non-amenability: existence of non-cyclic free subgroups.

**Another definition. (Gromov):**



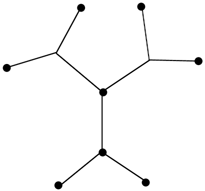
**Question. (Beatles)** Why can't we do it in the road?
**Answer:** Because $\mathbb{R}$ is amenable.

# von Neumann's problem: history

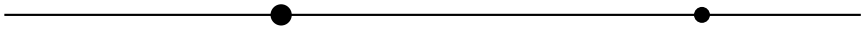
**Hausdorff, Banach, Tarski:** One can cut a ball into several pieces and move them to assemble two balls of the same size.
**von Neumann:** The reason is that the group of isometries of the ball is not *amenable*, i.e. there is no finitely additive left invariant probability measure on the set of all subsets of the group.

The reason for non-amenability: existence of non-cyclic free subgroups.

**Another definition. (Gromov):**



**Question. (Beatles)** Why can't we do it in the road?
**Answer:** Because $\mathbb{R}$ is amenable.



# von Neumann's problem

**Problem (von Neumann-Day, 50s)**

# von Neumann's problem

**Problem (von Neumann-Day, 50s)** Is there a non-amenable group without non-cyclic free subgroups?

# von Neumann's problem

**Problem (von Neumann-Day, 50s)** Is there a non-amenable group without non-cyclic free subgroups?

Solved in the 80s:

# von Neumann's problem

**Problem (von Neumann-Day, 50s)** Is there a non-amenable group without non-cyclic free subgroups?

Solved in the 80s: Olshanskii (Tarski monster),

# von Neumann's problem

**Problem (von Neumann-Day, 50s)** Is there a non-amenable group without non-cyclic free subgroups?

Solved in the 80s: Olshanskii (Tarski monster), Adian (the free Burnside groups of large enough exponent).

# Grigorchuk-Cohen problem

**Problem.**

# Grigorchuk-Cohen problem

**Problem.** Is there a finitely presented counterexample to von Neumann's problem?

# Grigorchuk-Cohen problem

**Problem.** Is there a finitely presented counterexample to von Neumann's problem?

**Theorem. (Olshanskii, S., Publ. IHES, 2002)**

# Grigorchuk-Cohen problem

**Problem.** Is there a finitely presented counterexample to von Neumann's problem?

**Theorem. (Olshanskii, S., Publ. IHES, 2002)** For every sufficiently large odd $n$,

# Grigorchuk-Cohen problem

**Problem.** Is there a finitely presented counterexample to von Neumann's problem?

**Theorem. (Olshanskii, S., Publ. IHES, 2002)** For every sufficiently large odd $n$, there exists a finitely presented group $\mathcal{G}$:

# Grigorchuk-Cohen problem

**Problem.** Is there a finitely presented counterexample to von Neumann's problem?

**Theorem. (Olshanskii, S., Publ. IHES, 2002)** For every sufficiently large odd $n$, there exists a finitely presented group $\mathcal{G}$:

1. $\mathcal{G}$ is an ascending HNN extension of a finitely generated infinite group $H$ of exponent $n$.

# Grigorchuk-Cohen problem

**Problem.** Is there a finitely presented counterexample to von Neumann's problem?

**Theorem. (Olshanskii, S., Publ. IHES, 2002)** For every sufficiently large odd $n$, there exists a finitely presented group $\mathcal{G}$:

1. $\mathcal{G}$ is an ascending HNN extension of a finitely generated infinite group $H$ of exponent $n$.
2. $H$ is an "almost" finitely presented bounded torsion group.

# Grigorchuk-Cohen problem

**Problem.** Is there a finitely presented counterexample to von Neumann's problem?

**Theorem. (Olshanskii, S., Publ. IHES, 2002)** For every sufficiently large odd $n$, there exists a finitely presented group $\mathcal{G}$:

1. $\mathcal{G}$ is an ascending HNN extension of a finitely generated infinite group $H$ of exponent $n$.
2. $H$ is an "almost" finitely presented bounded torsion group.
3. $\mathcal{G}$ contains a subgroup isomorphic to a free Burnside group of exponent $n$ with 2 generators.

# Grigorchuk-Cohen problem

**Problem.** Is there a finitely presented counterexample to von Neumann's problem?

**Theorem. (Olshanskii, S., Publ. IHES, 2002)** For every sufficiently large odd $n$, there exists a finitely presented group $\mathcal{G}$:

1. $\mathcal{G}$ is an ascending HNN extension of a finitely generated infinite group $H$ of exponent $n$.
2. $H$ is an "almost" finitely presented bounded torsion group.
3. $\mathcal{G}$ contains a subgroup isomorphic to a free Burnside group of exponent $n$ with 2 generators.
4. $\mathcal{G}$ is a non-amenable finitely presented group without free non-cyclic subgroups.

# Grigorchuk-Cohen problem

**Problem.** Is there a finitely presented counterexample to von Neumann's problem?

**Theorem. (Olshanskii, S., Publ. IHES, 2002)** For every sufficiently large odd $n$, there exists a finitely presented group $\mathcal{G}$:

1. $\mathcal{G}$ is an ascending HNN extension of a finitely generated infinite group $H$ of exponent $n$.
2. $H$ is an "almost" finitely presented bounded torsion group.
3. $\mathcal{G}$ contains a subgroup isomorphic to a free Burnside group of exponent $n$ with 2 generators.
4. $\mathcal{G}$ is a non-amenable finitely presented group without free non-cyclic subgroups.

The proof uses all the ideas mentioned above

# Grigorchuk-Cohen problem

**Problem.** Is there a finitely presented counterexample to von Neumann's problem?

**Theorem. (Olshanskii, S., Publ. IHES, 2002)** For every sufficiently large odd $n$, there exists a finitely presented group $\mathcal{G}$:

1. $\mathcal{G}$ is an ascending HNN extension of a finitely generated infinite group $H$ of exponent $n$.
2. $H$ is an "almost" finitely presented bounded torsion group.
3. $\mathcal{G}$ contains a subgroup isomorphic to a free Burnside group of exponent $n$ with 2 generators.
4. $\mathcal{G}$ is a non-amenable finitely presented group without free non-cyclic subgroups.

The proof uses all the ideas mentioned above plus the Olshanskii theory of graded diagrams.

# Grigorchuk-Cohen problem

**Problem.** Is there a finitely presented counterexample to von Neumann's problem?

**Theorem. (Olshanskii, S., Publ. IHES, 2002)** For every sufficiently large odd $n$, there exists a finitely presented group $\mathcal{G}$:

1. $\mathcal{G}$ is an ascending HNN extension of a finitely generated infinite group $H$ of exponent $n$.
2. $H$ is an "almost" finitely presented bounded torsion group.
3. $\mathcal{G}$ contains a subgroup isomorphic to a free Burnside group of exponent $n$ with 2 generators.
4. $\mathcal{G}$ is a non-amenable finitely presented group without free non-cyclic subgroups.

The proof uses all the ideas mentioned above plus the Olshanskii theory of graded diagrams.

**Problem.**

# Grigorchuk-Cohen problem

**Problem.** Is there a finitely presented counterexample to von Neumann's problem?

**Theorem. (Olshanskii, S., Publ. IHES, 2002)** For every sufficiently large odd $n$, there exists a finitely presented group $\mathcal{G}$:

1. $\mathcal{G}$ is an ascending HNN extension of a finitely generated infinite group $H$ of exponent $n$.
2. $H$ is an "almost" finitely presented bounded torsion group.
3. $\mathcal{G}$ contains a subgroup isomorphic to a free Burnside group of exponent $n$ with 2 generators.
4. $\mathcal{G}$ is a non-amenable finitely presented group without free non-cyclic subgroups.

The proof uses all the ideas mentioned above plus the Olshanskii theory of graded diagrams.

**Problem.** Is there a finitely presented torsion group?